Path-dependent BSDEs with jumps and their connection to PPIDEs

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Abstract

We study path-dependent backward stochastic differential equations (BSDEs) with jumps. In this context path-dependence of a BSDE is the dependence of the BSDE-terminal condition and the BSDE-generator of a path of a càdlàg process. We study the path-differentiability of BSDEs of this type and establish a connection to path-dependent PIDEs in terms of the existence of a viscosity solution and the respective Feynman-Kac theorem.

Keywords path-dependent backward stochastic differential equation; jump diffusion; path-dependent PIDE; functional Feynman-Kac theorem; path-differentiability; viscosity solution; functional Itô formula.

1 Introduction

Research on backward stochastic differential equations (BSDEs) started with Bismut Bismut (1973) who considered a linear BSDE for the purpose of solving an optimal stochastic control problem. Since then, BSDEs and many of their interesting applications in financial mathematics and stochastic optimization have been extensively studied in the literature.

Pardoux and Peng (1990) provided the first results on existence and uniqueness of solutions for nonlinear BSDEs with Lipschitz generator and their research has been extended in terms of BSDEs with Poisson jumps for a fixed terminal condition and Lipschitz coefficients by Tang and Li (1994) and with bounded random stopping time as its terminal time by Rong (1997). Becherer (2006) proved results on bounded solutions to backward stochastic equations driven by random measures. With regard to BSDEs with jumps and their Malliavin differentiability we refer the interested reader to Delong and Imkeller (2010) and the excellent monograph in Delong (2013).

The connections of BSDEs to financial mathematics are studied in El Karoui et al. (1997), Rong (2005), Carmona (2008), Kromer and Overbeck (2014) and many others. These include applications of BSDEs in non-linear pricing theory and in dynamic risk measurement.

In this paper we are primarily interested in a generalization of the theory of BSDEs with jumps that allows for the terminal condition and the generator of the BSDE to be a function of a path of a càdlàg process. We will study existence

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and uniqueness of solutions and path-differentiability of BSDEs of this type. Furthermore, we will derive a connection between path-dependent BSDEs with jumps and a certain path-dependent partial integro-differential equation (PPIDE) that arises in terms of the existence of a viscosity solution and the corresponding functional Feynman-Kac theorem. This approach requires functional derivatives and a functional Itô formula that was introduced for continuous processes in an insightful paper by Dupire (2009), and later on was extended to càdlàg semimartingales by Cont and Fournié (2010) and Levental et al. (2013).

The connection of BSDEs to quasi-linear partial differential equations (PDEs) was established in the classical case of continuous BSDEs by Peng (1991) and Pardoux and Peng (1992). This results were extended by Ma et al. (1994) and Pardoux and Tang (1999) for FBSDEs and quasi-linear PDEs. Recently Peng and Wang (2016) studied this connection for path-dependent continuous BSDEs and path-dependent PDEs. Keller (2016) extended the results of viscosity solutions for path-dependent PDEs introduced by Ekren et al. (2014) to path-dependent integro-differential equations. Barles et al. (1997) considered BSDEs with jumps and proved that under certain conditions the solution of such a BSDE provides a viscosity solution to a PIDE, where the definition of a viscosity solution is different from Keller (2016).

We extend results ofPeng and Wang (2016) to the case of path-dependent BSDEs with jumps and study the corresponding functional Feynman-Kac theorem that leads to a path-dependent PIDE. Independently to our work, Wang (2015) received similar results about path-dependent BSDEs and their connection to nonlinear PPIDEs. We show that this PPIDE establishes a connection to credit value adjustment (CVA), an important current topic in mathematical finance, see for instance Burgard and Kjaer (2011), Crépey (2012), Henry-Labordère (2012) and Piterbarg (2010). It provides an interpretation of the solution to the CVA-problem in terms of solutions to a specific path-dependent BSDE.

The outline of the paper is as follows. In Section 2 we present the necessary notation and provide the required definitions. In particular, we introduce Dupire's functional derivatives, the functional Itô formula and the path-dependent BSDE with jumps that is studied throughout the paper. In Section 3 we provide the existence and uniqueness result for the respective BSDE under consideration. Furthermore, we study the respective comparison theorem. In Section 5 we consider the path-differentiability of our BSDE in terms of Dupire's functional derivatives. In Section 6 we investigate the existence of a viscosity solution to a nonlinear path-dependent integro-differential equation. Finally, in Section 7 we establish a connection between path-dependent BSDEs with jumps and the corresponding path-dependent PIDEs in terms of a functional Feynman-Kac theorem.

2 Preliminaries

Let us consider a probability space $(\Omega,\mathscr{F},\mathbb{P})$ with filtration $(\mathscr{F}_s^t)_{s\in[t,T]}$ defined by $\mathscr{F}_s^t=\sigma\{\sigma(W(r)-W(t);t\le r\le s)\vee\mathscr{N}\}$, for any $0\le t\le s\le T$, where $(W(s))_{s\in[0,T]}$ is a d-dimensional Brownian motion and $\mathscr{N}$ the $\mathbb{P}$-null set. We denote by $\mathscr{F}_r$ the filtration $\mathscr{F}_r^0$ and by $\mathscr{F}^t$ the filtration $\mathscr{F}_T^t$. Let $T>0$ be fixed.

We denote by $D([0,T],\mathbb{R}^d)$ the space of all càdlàg $\mathbb{R}^d$-valued functions on $[0,T]$ endowed with the supremum metric, i.e.

$$\|\gamma\|:=\sup_{s\in[0,T]}|\gamma(s)|,\quad \|\gamma_t-\bar{\gamma}_{\bar{t}}\|_D:=\|\gamma_t-\bar{\gamma}_{\bar{t}}\|_{D(t,\bar{t})}=\|\gamma_t-\bar{\gamma}_{\bar{t}}\|+|t-\bar{t}|=\sup_{s\in[0,T]}|\gamma_t(s)-\bar{\gamma}_{\bar{t}}(s)|+|t-\bar{t}|,$$

for $\gamma_t,\bar{\gamma}_{\bar{t}}\in D([0,T],\mathbb{R}^d)$, $t,\bar{t}\in[0,T]$. For each $\gamma\in D([0,T],\mathbb{R}^d)$ the value of γ at $s\in[0,T]$ is $\gamma(s)$ and we define the path of γ stopped at $t\in[0,T]$, denoted by γ_t, as $\gamma_t(u)=\gamma(u\wedge t)=\gamma(u)\mathbb{1}_{[0,t)}(u)+\gamma(t)\mathbb{1}_{[t,T]}(u)$ for all $u\in[0,T]$. The path with a shift by $h\in\mathbb{R}^d$ in t is denoted by $\gamma_t^h(u)=\gamma(u\wedge t)+h\mathbb{1}_{[t,T]}(u)$, for $u\in[0,T]$.

For any $\gamma\in D([0,T],\mathbb{R}^d)$ and a fixed $t\in[0,T]$ we are looking for a unique solution to the following path-dependent BSDE

$$Y_{\gamma_t}(s)=\Phi(X_{\gamma_t,T})+\int_s^T f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))dr-\int_s^T Z_{\gamma_t}(r)dW(r)-\int_s^T\int_{\mathbb{R}^l}U_{\gamma_t}(r,x)\tilde{N}(dr,dx),\qquad(2.1)$$

with

$$X_{\gamma_t,s-}(u):=\gamma(u)\mathbb{1}_{[0,t)}(u)+(\gamma(t)+X(u)-X(t))\mathbb{1}_{[t,s)}(u)+(\gamma(t)+X(s)-X(t))\mathbb{1}_{[s,T]}(u),\ u\in[0,T],\qquad(2.2)$$

where $\tilde{N}(\omega,dt,dx)=N(\omega,dt,dx)-\nu(\omega,dt,dx)$ is a compensated random measure on the probability space and X a $\mathbb{R}^d$-valued adapted càdlàg process. In the following we will use the notation of Delong (2013), Levental et al. (2013) and Peng and Wang (2016).

- Let $\mathscr{M}(\mathbb{R}^l;\mathbb{R}^m)$ be the space of all measurable functions from $\mathbb{R}^l$ to $\mathbb{R}^m$,
- $L^2_\mu(\mathbb{R}^l;\mathbb{R}^m)$ denotes the space of measurable functions $\varphi:\mathbb{R}^l\to\mathbb{R}^m$ satisfying $\int_{\mathbb{R}^l}|\varphi(x)|^2\mu(dx)<\infty$,
- Let $L^2(D([0,T],\mathbb{R}^d);\mathbb{R}^m)=\{\psi:\Omega\times D([0,T],\mathbb{R}^d)\to\mathbb{R}^m|\psi(\cdot)\text{ is measurable and }\mathbb{E}[|\psi(\cdot)|^2]<\infty\}$.
- Similar, we define $L^2_{\mathscr{F}}(t,T):=\{h:\Omega\times[t,T]\to\mathbb{R}^m|h(s)\,\text{is}\,\mathscr{F}^t_s\text{-adapted}, \mathbb{E}[\int_t^T|h(s)|^2dr]<\infty\}$.
- We denote by $S^2[t,T]$ the space for process Y with

$$S^2[t,T]:=\Big\{Y:\Omega\times[t,T]\to\mathbb{R}^m:(Y(s))_{s\in[t,T]}\text{ is a càdlàg }\mathscr{F}^t_s\text{-adapted process, }\mathbb{E}\Big[\sup_{s\in[t,T]}|Y(s)|^2\Big]<\infty\Big\}.$$

- For process Z we introduce the space

$$M^2[t,T]:=\Big\{Z:\Omega\times[t,T]\to\mathbb{R}^{m\times d}:(Z(s))_{s\in[t,T]}\text{ is predictable, }\mathbb{E}\Big[\int_t^T\|Z(s)\|^2ds\Big]<\infty\Big\},$$

  where we denote $\|z\|=\sqrt{tr(zz^*)}$ for $z\in\mathbb{R}^{m\times d}$.
- The space for process U is denoted by

$$L^2_N(t,T)=\left\{U:\Omega\times[t,T]\times\mathbb{R}^l\to\mathbb{R}^m\,\middle|\,(U(s,x))_{s\in[t,T],x\in\mathbb{R}^l}\text{ is predictable,}\,\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^l}|U(s,x)|^2\nu(ds,dx)\right]<\infty\right\},$$

  where we integrate with respect to the predictable compensator of the random measure N.
- Furthermore, we define the space $\mathbb{H}^2[t,T]:=S^2[t,T]\times M^2[t,T]\times L^2_N(t,T)$, and equip it with the norm

$$\|(Y,Z,U)\|_{\mathbb{H}^2[t,T]}=\left(\mathbb{E}\left[\sup_{s\in[t,T]}|Y(s)|^2+\int_t^T\|Z(r)\|^2dr+\int_t^T\int_{\mathbb{R}^l}|U(r,x)|^2\nu(dr,dx)\right]\right)^{\frac{1}{2}}.$$

We will assume that the following conditions on the random measure and the corresponding compensator hold.

(RM) The random measure N is an integer-valued random measure with compensator

$$\nu(\omega,dt,dx)=Q(\omega,t,dx)\eta(\omega,t)dt,$$

where $\eta:\Omega\times[0,T]\to[0,\infty)$ is a predictable process, and Q is a kernel satisfying

$$\int_0^T\int_{\mathbb{R}^l}|x|^2Q(\omega,t,dx)\eta(\omega,t)dt<\infty.$$

Note that for a given compensator ν of a random measure N the compensated random measure can be defined by $\tilde N(\omega,dt,dx)=N(\omega,dt,dx)-\nu(\omega,dt,dx)$. The compensator is uniquely determined, see Theorem 11.6 in He et al. (1992).

(PR) Any $\mathscr{F}$-local martingale M has the representation

$$M(s)=M(t)+\int_t^sZ(r)dW(r)+\int_t^s\int_{\mathbb{R}^l}U(r,x)\tilde N(dr,dx),\quad s\in[t,T],\tag{2.3}$$

where Z and U are $\mathscr{F}$-predictable processes integrable with respect to W and $\tilde N$. Moreover, any square integrable $\mathscr{F}$-martingale M has the unique representation

$$M(s)=M(t)+\int_t^sZ(r)dW(r)+\int_t^s\int_{\mathbb{R}^l}U(r,x)\tilde N(dr,dx),\quad s\in[t,T],\tag{2.4}$$

where $(Z,U)\in M^2[t,T]\times L^2_N(t,T)$. For more details see Sect. III.4 in Jacod and Shiryaev (2003) or Sect. XIII.2 in He et al. (1992).

Definition 2.1. *$F : D([0,T],\mathbb{R}^d) \to \mathbb{R}^m$ is continuous at $\gamma \in D([0,T],\mathbb{R}^d)$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\bar{\gamma} \in D([0,T],\mathbb{R}^d)$ satisfying $\|\gamma - \bar{\gamma}\|_D < \delta$ it is $|F(\gamma) - F(\bar{\gamma})| < \varepsilon$. F is continuous if it is continuous at each $\gamma \in D([0,T],\mathbb{R}^d)$.*

Definition 2.2. *Let $F : D([0,T],\mathbb{R}^d) \to \mathbb{R}^m$ and $\{e_i\}_{i=1,\ldots,d}$ the canonical basis of $\mathbb{R}^d$. For $\gamma \in D([0,T],\mathbb{R}^d)$ we denote by $D_i^1 F(\gamma;[t,T])$, $i = 1,\ldots,d$, the derivative of F at γ in the direction of the $\mathbb{R}^d$-valued process $\mathbb{1}_{[t,T]}e_i$, which means*

$$D_i^1 F(\gamma;[t,T]) = \lim_{h\to 0} \frac{F(\gamma + h\mathbb{1}_{[t,T]}e_i) - F(\gamma)}{h}.$$

By

$$D_{ij}^2 F(\gamma;[t,T]) = \lim_{h\to 0} \frac{D_j^1 F(\gamma + h\mathbb{1}_{[t,T]}e_i;[t,T]) - D_j^1 F(\gamma;[t,T])}{h}$$

we denote the second-order derivative in the direction $1_{[t,T]}e_i$ and $1_{[t,T]}e_j$, for $i,j = 1,\ldots,d$. Notice that the derivative $D_i^1 F(\gamma;[t,T])$ is in $\mathbb{R}^m$ and $D^1 F(\gamma;[t,T]) \in \mathbb{R}^{m\times d}$. Furthermore, the second-order derivative $D_{ij}^2 F(\gamma;[t,T])$ is in $\mathbb{R}^m$ and $D^2 F(\gamma;[t,T])$ is a tensor. F is said to be of class $C^1(D([0,T],\mathbb{R}^d))$ if the derivatives $D_i^1 F(\gamma;[t,T])$ exist and are continuous. The class $C^2(D([0,T],\mathbb{R}^d))$ can be defined similarly.

These derivatives from Levental et al. (2013) are consistent with the derivatives from Dupire (2009). For a comparison of these approaches we refer the reader to Appendix A.1 in Levental et al. (2013). Note that the derivatives are functions of $(\gamma,t) \in D([0,T],\mathbb{R}^d) \times [0,T]$.

In the following we will need the functional Itô formula for càdlàg semimartingales from Levental et al. (2013).

Theorem 2.3. *Let X be a d-dimensional càdlàg semimartingale. Let $F : D([0,T],\mathbb{R}^d) \to \mathbb{R}$ be a functional which is continuous and $D_i^1 F(x;[t,T])$, $D_{ij}^2 F(x;[t,T])$, $i,j = 1,\ldots,d$, exist and are continuous in x and t. Then*

$$\begin{aligned} F(X_t) &= F(X_0) + \sum_{i=1}^d \int_0^t D_i^1 F(X_{s-};[s,T])\, dX^i(s) + \frac{1}{2}\sum_{i,j=1}^d \int_0^t D_{ij}^2 F(X_{s-};[s,T])\, d\langle X^{i,c}(s), X^{i,c}(s)\rangle \\ &\quad + \sum_{s\le t}\left[F(X_s) - F(X_{s-}) - \sum_{i=1}^d D_i^1 F(X_{s-};[s,T])\,\Delta X^i(s)\right], \end{aligned}$$

where X^c is the continuous part of the semimartingale X and $\Delta X^i(s) = X^i(s) - X^i(s-)$.

Remark 2.4. *If we construct $\tilde{F} : D([0,T],\mathbb{R}_+ \times \mathbb{R}^d) \to \mathbb{R}$ such that $\tilde{F}(x^0,x) = F(x^0(T)\wedge T, x)$ for a functional $F : [0,T]\times D([0,T],\mathbb{R}^d) \to \mathbb{R}$ it follows for $x^0 \in D([0,T],\mathbb{R}_+)$ with $x^0(s) = s$ the time and path dependent Itô formula for càdlàg semimartingales*

$$\begin{aligned} F(t,X_t) &= F(0,X_0) + \int_0^t \frac{\partial}{\partial s} F(s,X_{s-})ds + \sum_{i=1}^d \int_0^t D_i^1 F(s,X_{s-};[s,T])\, dX^i(s) \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_0^t D_{ij}^2 F(s,X_{s-};[s,T])\, d\langle X^{i,c}(s), X^{i,c}(s)\rangle + \sum_{s\le t}\left[F(s,X_s) - F(s,X_{s-}) - \sum_{i=1}^d D_i^1 F(s,X_{s-};[s,T])\,\Delta X^i(s)\right]. \end{aligned}$$

Additionally we also refer to the version of the functional Itô formula in Cont and Fournié (2010).

3 Adapted solution to the path-dependent BSDE with jumps

In this section we will study path-dependent BSDEs with jumps (2.1) for any given $\gamma \in D([0,T],\mathbb{R}^d)$, a fixed $t \in [0,T]$ and a known $\mathbb{R}^d$-valued adapted càdlàg process X. Notice that $X_{\gamma_t,r}$ is in $D([0,T],\mathbb{R}^d)$ for all $r \in [t,T]$. The function $f : \Omega \times [t,T] \times D([0,T],\mathbb{R}^d) \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathscr{M}(\mathbb{R}^d;\mathbb{R}^m) \to \mathbb{R}^m$ is called the generator of the BSDE (2.1). A solution to such a path-dependent BSDE is a triple $(Y_{\gamma_t}, Z_{\gamma_t}, U_{\gamma_t})$, where $Y_{\gamma_t} = (Y_{\gamma_t}(s); s \in [t,T])$ is $\mathscr{F}$-adapted, $Z_{\gamma_t} = (Z_{\gamma_t}(s); s \in [t,T])$ and $U_{\gamma_t} = (U_{\gamma_t}(s,x); s \in [t,T], x \in \mathbb{R}^l)$ are predictable processes. Z_{γ_t} and U_{γ_t} are called the control processes of BSDE (2.1).

We now assume that

(A1) $\Phi \in L^2\big(D([0,T],\mathbb{R}^d);\mathbb{R}^m\big)$,

(A2) the generator $f:\Omega\times[t,T]\times D([0,T],\mathbb{R}^d)\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathscr{M}(\mathbb{R}^l;\mathbb{R}^m)\to\mathbb{R}^m$ is predictable and Lipschitz continuous in the sense that

$$|f(\omega,s,\gamma,y,z,u)-f(\omega,s,\bar\gamma,\bar y,\bar z,\bar u)|\le K(s)\Big[\|\gamma-\bar\gamma\|_D+|y-\bar y|+\|z-\bar z\|+\int_{\mathbb{R}^l}|u(x)-\bar u(x)|Q(\omega,t,dx)\eta(\omega,t)\Big], \tag{3.1}$$

a.e. $(\omega,s)\in\Omega\times[t,T]$ for all $(\gamma,y,z,u),(\bar\gamma,\bar y,\bar z,\bar u)\in D([0,T],\mathbb{R}^d)\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2_{Q(\omega)}(\mathbb{R}^l;\mathbb{R}^m)$ and with $K:=\sup_{s\in[t,T]}|K(s)|<\infty$,

(A3) $\mathbb{E}[\int_t^T|f_0(r,\gamma_{r-})|^2dr]<\infty$, with $f_0(r,\gamma_r):=f(r,\gamma_r,0,0,0)$.

Lemma 3.1. *Assume that (Φ,f) and $(\bar\Phi,\bar f)$ satisfy (A1)-(A3), $t\in[0,T]$. For any $\gamma,\bar\gamma\in D([0,T],\mathbb{R}^d)$ and $t,\bar t\in[0,T]$, let $(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t})\in\mathbb{H}^2[t,T]$ and $(\bar Y_{\bar\gamma_{\bar t}},\bar Z_{\bar\gamma_{\bar t}},\bar U_{\bar\gamma_{\bar t}})\in\mathbb{H}^2[\bar t,T]$ be solutions to the path-dependent BSDE (2.1) with (Φ,f) and $(\bar\Phi,\bar f)$. We have the estimates*

$$\begin{aligned}&\big\|(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t})-(\bar Y_{\bar\gamma_{\bar t}},\bar Z_{\bar\gamma_{\bar t}},\bar U_{\bar\gamma_{\bar t}})\big\|^2_{\mathbb{H}^2[t\vee\bar t,T]}\\&\le C\Bigg(\mathbb{E}\left[|\Phi(X_{\gamma_t,T})-\bar\Phi(X_{\bar\gamma_{\bar t},T})|^2\right]\\&+\mathbb{E}\left[\int_{t\vee\bar t}^T|f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar f(r,X_{\bar\gamma_{\bar t},r-},\bar Y_{\bar\gamma_{\bar t}}(r-),\bar Z_{\bar\gamma_{\bar t}}(r),\bar U_{\bar\gamma_{\bar t}}(r))|^2dr\right]\Bigg)\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\big\|(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t})-(\bar Y_{\bar\gamma_{\bar t}},\bar Z_{\bar\gamma_{\bar t}},\bar U_{\bar\gamma_{\bar t}})\big\|^2_{\mathbb{H}^2[t\vee\bar t,T]}\\&\le C\Bigg(\mathbb{E}\left[|\Phi(X_{\gamma_t,T})-\bar\Phi(X_{\bar\gamma_{\bar t},T})|^2\right]\\&+\mathbb{E}\left[\int_{t\vee\bar t}^T|f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar f(r,X_{\bar\gamma_{\bar t},r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))|^2dr\right]\Bigg).\end{aligned} \tag{3.3}$$

Proof. This proof uses techniques of the proof of Lemma 3.1.1 in Delong (2013). Since $(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t}),(\bar Y_{\bar\gamma_{\bar t}},\bar Z_{\bar\gamma_{\bar t}},\bar U_{\bar\gamma_{\bar t}})\in\mathbb{H}^2[t,T]$ are solutions to the path-dependent BSDE (2.1) with (Φ,f) and $(\bar\Phi,\bar f)$ we get

$$\begin{aligned}Y_{\gamma_t}(s)-\bar Y_{\bar\gamma_{\bar t}}(s)&=\Phi(X_{\gamma_t,T})-\bar\Phi(X_{\bar\gamma_{\bar t},T})+\int_s^T f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar f(r,X_{\bar\gamma_{\bar t},r-},\bar Y_{\bar\gamma_{\bar t}}(r-),\bar Z_{\bar\gamma_{\bar t}}(r),\bar U_{\bar\gamma_{\bar t}}(r))dr\\&-\int_s^T Z_{\gamma_t}(r)-\bar Z_{\bar\gamma_{\bar t}}(r)dW(r)-\int_s^T\int_{\mathbb{R}^l}U_{\gamma_t}(r,x)-\bar U_{\bar\gamma_{\bar t}}(r,x)\tilde N(dr,dx),\quad s\in[t,T].\end{aligned}$$

Applying Itô's formula for semimartingales to $|Y_{\gamma_t}(s)-\bar Y_{\bar\gamma_{\bar t}}(s)|^2$ and sorting the terms yields

$$\begin{aligned}&|\Phi(X_{\gamma_t,T})-\bar\Phi(X_{\bar\gamma_{\bar t},T})|^2-|Y_{\gamma_t}(s)-\bar Y_{\bar\gamma_{\bar t}}(s)|^2-\int_s^T\|Z_{\gamma_t}(r)-\bar Z_{\bar\gamma_{\bar t}}(r)\|^2dr-\int_s^T\int_{\mathbb{R}^l}|U_{\gamma_t}(r,x)-\bar U_{\bar\gamma_{\bar t}}(r,x)|^2\nu(dr,dx)\\&+2\int_s^T\langle Y_{\gamma_t}(r-)-\bar Y_{\bar\gamma_{\bar t}}(r-),f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar f(r,X_{\bar\gamma_{\bar t},r-},\bar Y_{\bar\gamma_{\bar t}}(r-),\bar Z_{\bar\gamma_{\bar t}}(r),\bar U_{\bar\gamma_{\bar t}}(r))\rangle dr\\&=2\int_s^T\langle Y_{\gamma_t}(r-)-\bar Y_{\bar\gamma_{\bar t}}(r-),Z_{\gamma_t}(r)-\bar Z_{\bar\gamma_{\bar t}}(r)\rangle dW(r)+2\int_s^T\int_{\mathbb{R}^l}\langle Y_{\gamma_t}(r-)-\bar Y_{\bar\gamma_{\bar t}}(r-),U_{\gamma_t}(r,x)-\bar U_{\bar\gamma_{\bar t}}(r,x)\rangle\tilde N(dr,dx)\\&+\int_s^T\int_{\mathbb{R}^l}|U_{\gamma_t}(r,x)-\bar U_{\bar\gamma_{\bar t}}(r,x)|^2\tilde N(dr,dx).\end{aligned} \tag{3.4}$$

With

$$2|\langle u,v\rangle|\le\frac{1}{\alpha}|u|^2+\alpha|v|^2,\qquad\alpha>0, \tag{3.5}$$

it follows

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|2\int_s^T \langle Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-), f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar{f}(r,X_{\bar{\gamma}_t,r-},\bar{Y}_{\bar{\gamma}_t}(r-),\bar{Z}_{\bar{\gamma}_t}(r),\bar{U}_{\bar{\gamma}_t}(r))\rangle dr\right|\right]$$
$$\leq \frac{T-t}{\alpha}\mathbb{E}\left[\sup_{s\in[t,T]}|Y_{\gamma_t}(s)-\bar{Y}_{\bar{\gamma}_t}(s)|^2\right]$$
$$+\alpha\mathbb{E}\left[\int_t^T |f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar{f}(r,X_{\bar{\gamma}_t,r-},\bar{Y}_{\bar{\gamma}_t}(r-),\bar{Z}_{\bar{\gamma}_t}(r),\bar{U}_{\bar{\gamma}_t}(r))|^2 dr\right]. \tag{3.6}$$

Since $(Y_{\gamma_t},U_{\gamma_t},Z_{\gamma_t})$, $(\bar{Y}_{\bar{\gamma}_t},\bar{Z}_{\bar{\gamma}_t},\bar{U}_{\bar{\gamma}_t})\in\mathbb{H}^2[t,T]$ and with the assumptions (A2) and (A3) for the generators f and $\bar{f}$ we get

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|2\int_s^T \langle Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-), f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar{f}(r,X_{\bar{\gamma}_t,r-},\bar{Y}_{\bar{\gamma}_t}(r-),\bar{Z}_{\bar{\gamma}_t}(r),\bar{U}_{\bar{\gamma}_t}(r))\rangle dr\right|\right]$$
$$\leq \frac{T-t}{\alpha}\mathbb{E}\left[\sup_{s\in[t,T]}|Y_{\gamma_t}(s)-\bar{Y}_{\bar{\gamma}_t}(s)|^2\right]+4\alpha K\left\{(T-t)\mathbb{E}\left[\sup_{s\in[t,T]}|Y_{\gamma_t}(s)|^2\right]+\mathbb{E}\left[\int_t^T\|Z_{\gamma_t}(r)\|^2dr\right]\right.$$
$$\left.+\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^l}|U_{\gamma_t}(r,x)|^2\nu(dr,dx)\right]\right\}+4\alpha\mathbb{E}\Big[\int_t^T|f_0(r,X_{\gamma_t,r-})|^2dr\Big]+4\alpha K\left\{(T-t)\mathbb{E}\left[\sup_{s\in[t,T]}|\bar{Y}_{\bar{\gamma}_t}(s)|^2\right]\right.$$
$$\left.+\mathbb{E}\left[\int_t^T\|\bar{Z}_{\bar{\gamma}_t}(r)\|^2dr\right]+\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^l}|\bar{U}_{\bar{\gamma}_t}(r,x)|^2\nu(dr,dx)\right]\right\}+4\alpha\mathbb{E}\left[\int_t^T|\bar{f}_0(r,X_{\bar{\gamma}_t,r-})|^2dr\right]<\infty.$$

Hence, we obtain with equation (3.4)

$$\mathbb{E}\Bigg[\sup_{s\in[t,T]}\Bigg|2\int_s^T\langle Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-),Z_{\gamma_t}(r)-\bar{Z}_{\bar{\gamma}_t}(r)\rangle dW(r)+2\int_s^T\int_{\mathbb{R}^l}\langle Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-),U_{\gamma_t}(r,x)-\bar{U}_{\bar{\gamma}_t}(r,x)\rangle\tilde{N}(dr,dx)$$
$$+\int_s^T\int_{\mathbb{R}^l}|U_{\gamma_t}(r,x)-\bar{U}_{\bar{\gamma}_t}(r,x)|^2\tilde{N}(dr,dx)\Bigg|\Bigg]<\infty.$$

Therefore, the right hand side of equation (3.4) is a uniformly integrable martingale, see Theorem I.51 in Protter (2004), and we get

$$\mathbb{E}\left[|Y_{\gamma_t}(s)-\bar{Y}_{\bar{\gamma}_t}(s)|^2\right]+\mathbb{E}\left[\int_s^T\|Z_{\gamma_t}(r)-\bar{Z}_{\bar{\gamma}_t}(r)\|^2dr\right]+\mathbb{E}\left[\int_s^T\int_{\mathbb{R}^l}|U_{\gamma_t}(r,x)-\bar{U}_{\bar{\gamma}_t}(r,x)|^2\nu(dr,dx)\right]$$
$$\leq\mathbb{E}\left[|\Phi(X_{\gamma_t,T})-\bar{\Phi}(X_{\bar{\gamma}_t,T})|^2\right]$$
$$+2\mathbb{E}\left[\left|\int_s^T\langle Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-),f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar{f}(r,X_{\bar{\gamma}_t,r-},\bar{Y}_{\bar{\gamma}_t}(r-),\bar{Z}_{\bar{\gamma}_t}(r),\bar{U}_{\bar{\gamma}_t}(r))\rangle dr\right|\right]. \tag{3.7}$$

For $s=t$ and with the help of inequality (3.5) Gronwall's lemma provides

$$\mathbb{E}\left[\int_t^T\|Z_{\gamma_t}(r)-\bar{Z}_{\bar{\gamma}_t}(r)\|^2dr\right]+\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^l}|U_{\gamma_t}(r,x)-\bar{U}_{\bar{\gamma}_t}(r,x)|^2\nu(dr,dx)\right]$$
$$\leq K\Bigg(\mathbb{E}\left[|\Phi(X_{\gamma_t,T})-\bar{\Phi}(X_{\bar{\gamma}_t,T})|^2\right]$$
$$+\mathbb{E}\left[\int_t^T|f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar{f}(r,X_{\bar{\gamma}_t,r-},\bar{Y}_{\bar{\gamma}_t}(r-),\bar{Z}_{\bar{\gamma}_t}(r),\bar{U}_{\bar{\gamma}_t}(r))|^2dr\right]\Bigg). \tag{3.8}$$

Besides, with (3.4) it follows

$$\begin{aligned}
&\sup_{s\in[t,T]} |Y_{\gamma_t}(s)-\bar{Y}_{\bar{\gamma}_t}(s)|^2 \\
&\quad\leq |\Phi(X_{\gamma_t,T})-\bar{\Phi}(X_{\bar{\gamma}_t,T})|^2 + 2\sup_{s\in[t,T]}\left|\int_s^T \langle Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-), Z_{\gamma_t}(r)-\bar{Z}_{\bar{\gamma}_t}(r)\rangle dW(r)\right| \\
&\qquad+\sup_{s\in[t,T]}\left|2\int_s^T \langle Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-), f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar{f}(r,X_{\bar{\gamma}_t,r-},\bar{Y}_{\bar{\gamma}_t}(r-),\bar{Z}_{\bar{\gamma}_t}(r),\bar{U}_{\bar{\gamma}_t}(r))\rangle dr\right| \\
&\qquad+2\sup_{s\in[t,T]}\left|\int_s^T\int_{\mathbb{R}^l} \langle Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-), U_{\gamma_t}(r,x)-\bar{U}_{\bar{\gamma}_t}(r,x)\rangle \tilde{N}(dr,dx)\right|.
\end{aligned}$$

Inequality (3.5) yields

$$\begin{aligned}
&\sup_{s\in[t,T]}\left|2\int_s^T \langle Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-), f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar{f}(r,X_{\bar{\gamma}_t,r-},\bar{Y}_{\bar{\gamma}_t}(r-),\bar{Z}_{\bar{\gamma}_t}(r),\bar{U}_{\bar{\gamma}_t}(r))\rangle dr\right| \\
&\leq \frac{1}{\rho}\int_t^T |Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-)|^2 dr + \rho\int_t^T |f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar{f}(r,X_{\bar{\gamma}_t,r-},\bar{Y}_{\bar{\gamma}_t}(r-),\bar{Z}_{\bar{\gamma}_t}(r),\bar{U}_{\bar{\gamma}_t}(r))|^2 dr.
\end{aligned}$$

With the Burkholder-Davis-Gundy inequality and inequality (3.5) it is for $\alpha > 0$

$$\begin{aligned}
&2\mathbb{E}\left[\sup_{s\in[t,T]}\left|\int_s^T \langle Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-), Z_{\gamma_t}(r)-\bar{Z}_{\bar{\gamma}_t}(r)\rangle dW(r)\right|\right] \\
&\leq 2K_1\mathbb{E}\left[\left\{\int_t^T |\langle Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-), Z_{\gamma_t}(r)-\bar{Z}_{\bar{\gamma}_t}(r)\rangle|^2 dr\right\}^{\frac{1}{2}}\right] \\
&\leq K_1\left(\frac{1}{\alpha}\mathbb{E}\left[\sup_{s\in[t,T]}|Y_{\gamma_t}(s)-\bar{Y}_{\bar{\gamma}_t}(s)|^2\right] + \alpha\mathbb{E}\left[\int_t^T \|Z_{\gamma_t}(r)-\bar{Z}_{\bar{\gamma}_t}(r)\|^2 dr\right]\right).
\end{aligned}$$

Similar, we get with Theorem 2.3.3 in Delong (2013), the Burkholder-Davis-Gundy inequality and inequality (3.5)

$$\begin{aligned}
&2\mathbb{E}\left[\sup_{s\in[t,T]}\left|\int_s^T\int_{\mathbb{R}^l} \langle Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-), U_{\gamma_t}(r,x)-\bar{U}_{\bar{\gamma}_t}(r,x)\rangle \tilde{N}(dr,dx)\right|\right] \\
&\leq 2K_2\left(\frac{1}{\alpha}\mathbb{E}\left[\sup_{s\in[t,T]}|Y_{\gamma_t}(s)-\bar{Y}_{\bar{\gamma}_t}(s)|^2\right] + \alpha\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^l} |U_{\gamma_t}(r,x)-\bar{U}_{\bar{\gamma}_t}(r,x)|^2 \nu(dr,dx)\right]\right).
\end{aligned}$$

Together it follows

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,T]}|Y_{\gamma_t}(s)-\bar{Y}_{\bar{\gamma}_t}(s)|^2\right] \\
&\quad\leq \mathbb{E}\left[|\Phi(X_{\gamma_t,T})-\bar{\Phi}(X_{\bar{\gamma}_t,T})|^2\right] + \frac{K_1+2K_2}{\alpha}\mathbb{E}\left[\sup_{s\in[t,T]}|Y_{\gamma_t}(s)-\bar{Y}_{\bar{\gamma}_t}(s)|^2\right] + \alpha K_1\mathbb{E}\left[\int_t^T \|Z_{\gamma_t}(r)-\bar{Z}_{\bar{\gamma}_t}(r)\|^2 dr\right] \\
&\qquad+2\mathbb{E}\left[\int_t^T |\langle Y_{\gamma_t}(r-)-\bar{Y}_{\bar{\gamma}_t}(r-), f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar{f}(r,X_{\bar{\gamma}_t,r-},\bar{Y}_{\bar{\gamma}_t}(r-),\bar{Z}_{\bar{\gamma}_t}(r),\bar{U}_{\bar{\gamma}_t}(r))\rangle| dr\right] \\
&\qquad+2\alpha K_2\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^l} |U_{\gamma_t}(r,x)-\bar{U}_{\bar{\gamma}_t}(r,x)|^2 \nu(dr,dx)\right] \qquad (3.9)
\end{aligned}$$

and, for $\alpha > K_1 + 2K_2$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,T]}|Y_{\gamma_t}(s)-\bar{Y}_{\bar{\gamma}_t}(s)|^2\right] \\
&\quad \le \tilde{C}\Bigg(\mathbb{E}\left[|\Phi(X_{\gamma_t,T})-\bar{\Phi}(X_{\bar{\gamma}_t,T})|^2\right]+\alpha K_1\mathbb{E}\left[\int_t^T\|Z_{\gamma_t}(r)-\bar{Z}_{\bar{\gamma}_t}(r)\|^2dr\right] \\
&\qquad +2\alpha K_2\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^l}|U_{\gamma_t}(r,x)-\bar{U}_{\bar{\gamma}_t}(r,x)|^2\nu(dr,dx)\right]+\frac{T-t}{\rho}\mathbb{E}\left[\sup_{s\in[t,T]}|Y_{\gamma_t}(s)-\bar{Y}_{\bar{\gamma}_t}(s)|^2\right] \\
&\qquad +\rho\mathbb{E}\left[\int_t^T|f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar{f}(r,X_{\bar{\gamma}_t,r-},\bar{Y}_{\bar{\gamma}_t}(r-),\bar{Z}_{\bar{\gamma}_t}(r),\bar{U}_{\bar{\gamma}_t}(r))|^2dr\right]\Bigg).
\end{aligned}$$

If we choose $\rho > T-t$ it follows with (3.8) and $\tilde{K} = \alpha\max\{K_1, 2K_2\}$

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,T]}|Y_{\gamma_t}(s)-\bar{Y}_{\bar{\gamma}_t}(s)|^2\right] \\
&\quad \le C\Bigg(\mathbb{E}\left[|\Phi(X_{\gamma_t,T})-\bar{\Phi}(X_{\bar{\gamma}_t,T})|^2\right] \\
&\qquad +\mathbb{E}\left[\int_t^T|f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar{f}(r,X_{\bar{\gamma}_t,r-},\bar{Y}_{\bar{\gamma}_t}(r-),\bar{Z}_{\bar{\gamma}_t}(r),\bar{U}_{\bar{\gamma}_t}(r))|^2dr\right]\Bigg),
\end{aligned}$$

which proves equation (3.2). Equation (3.2) can be proved quite similar. □

Corollary 3.2. *For solutions $(Y_{\gamma_t}, Z_{\gamma_t}, U_{\gamma_t}), (Y_{\bar{\gamma}_t}, Z_{\bar{\gamma}_t}, U_{\bar{\gamma}_t}) \in \mathbb{H}^2[t,T]$ of BSDE (2.1) with $\gamma, \bar{\gamma} \in D([0,T], \mathbb{R}^d)$ we get*

$$\left\|(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t})-(Y_{\bar{\gamma}_t},Z_{\bar{\gamma}_t},U_{\bar{\gamma}_t})\right\|^2_{\mathbb{H}^2[t,T]} \le C\mathbb{E}\left[|\Phi(X_{\gamma_t,T})-\Phi(X_{\bar{\gamma}_t,T})|^2+\|X_{\gamma_t,T}-X_{\bar{\gamma}_t,T}\|_D^2\right]. \tag{3.10}$$

Theorem 3.3. *Assume that (A1)-(A3) hold. For each $\gamma \in D([0,T], \mathbb{R}^d)$, $t \in [0,T]$, BSDE (2.1) has a unique solution $(Y_{\gamma_t}, Z_{\gamma_t}, U_{\gamma_t}) \in \mathbb{H}^2[t,T]$ with*

$$Y_{\gamma_t}(s) = \mathbb{E}\left[\Phi(X_{\gamma_t,T})+\int_s^T f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))dr\,\middle|\,\mathscr{F}_s^t\right]$$

and the following estimate holds

$$\left\|(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t})\right\|^2_{\mathbb{H}^2[t,T]} \le C\left(\mathbb{E}\left[|\Phi(X_{\gamma_t,T})|^2\right]+\mathbb{E}\left[\int_t^T|f_0(r,X_{\gamma_t,r-})|^2dr\right]\right). \tag{3.11}$$

Proof. Existence and uniqueness: The proof follows similar argumentations as in Pardoux and Peng (1992).

First step: We are looking for the solution to BSDE

$$Y_{\gamma_t}(s) = \Phi(X_{\gamma_t,T})+\int_s^T f(r,X_{\gamma_t,r-},\bar{Y}(r-),\bar{Z}(r),\bar{U}(r))dr-\int_s^T Z_{\gamma_t}(r)dW(r)-\int_s^T\int_{\mathbb{R}^l}U_{\gamma_t}(r,x)\tilde{N}(dr,dx), \quad s\in[t,T], \tag{3.12}$$

for given $(\bar{Y}, \bar{Z}, \bar{U}) \in \mathbb{H}^2[t,T]$. This BSDE admits a unique solution in $\mathbb{H}^2[t,T]$ which is explicitly given by

$$Y_{\gamma_t}(s) = \mathbb{E}\left[\Phi(X_{\gamma_t,T})+\int_s^T f(r,X_{\gamma_t,r-},\bar{Y}(r-),\bar{Z}(r),\bar{U}(r))dr\,\middle|\,\mathscr{F}_s^t\right]$$

and $(Z_{\gamma_t}, U_{\gamma_t}) \in M^2[t,T] \times L_N^2(t,T)$ are the unique processes given by the representation (PR) such that

$$\begin{aligned}
&\int_t^T Z_{\gamma_t}(r)dW(r)+\int_t^T\int_{\mathbb{R}^l}U_{\gamma_t}(r,x)\tilde{N}(dr,dx) \\
&= \Phi(X_{\gamma_t,T})+\int_t^T f(r,X_{\gamma_t,r-},\bar{Y}(r-),\bar{Z}(r),\bar{U}(r))dr-\mathbb{E}\left[\Phi(X_{\gamma_t,T})+\int_t^T f(r,X_{\gamma_t,r-},\bar{Y}(r-),\bar{Z}(r),\bar{U}(r))dr\right].
\end{aligned}$$

Second step: Now we consider BSDE

$$Y_{\gamma_t}(s) = \Phi(X_{\gamma_t,T}) + \int_s^T f(r, X_{\gamma_t,r-}, \bar{Y}(r-), Z_{\gamma_t}(r), U_{\gamma_t}(r))de - \int_s^T Z_{\gamma_t}(r)dW(r) - \int_s^T \int_{\mathbb{R}^l} U_{\gamma_t}(r,x)\tilde{N}(dr,dx), \quad (3.13)$$

for a given $\bar{Y} \in S^2[t,T]$, $s \in [t,T]$. First, we want to prove the uniqueness of the solution to BSDE (3.13). Let $(Y^1_{\gamma_t}, Z^1_{\gamma_t}, U^1_{\gamma_t})$ and $(Y^2_{\gamma_t}, Z^2_{\gamma_t}, U^2_{\gamma_t})$ be two solutions. With (3.7) it follows

$$\begin{aligned}
&\mathbb{E}\left[|Y^1_{\gamma_t}(s) - Y^2_{\gamma_t}(s)|^2\right] + \mathbb{E}\left[\int_s^T \|Z^1_{\gamma_t}(r) - Z^2_{\gamma_t}(r)\|^2 dr\right] + \mathbb{E}\left[\int_s^T \int_{\mathbb{R}^l} |U^1_{\gamma_t}(r,x) - U^2_{\gamma_t}(r,x)|^2 \nu(dr,dx)\right] \\
&\le 2\mathbb{E}\left[\int_s^T |\langle Y^1_{\gamma_t}(r-) - Y^2_{\gamma_t}(r-), f(r, X_{\gamma_t,r-}, \bar{Y}(r-), Z^1_{\gamma_t}(r), U^1_{\gamma_t}(r)) - f(r, X_{\gamma_t,r-}, \bar{Y}(r-), Z^2_{\gamma_t}(r), U^2_{\gamma_t}(r))\rangle| dr\right].
\end{aligned}$$

Inequality (3.5) with $\alpha = 2K$ and the Lipschitz condition of the generator f yield for $s \in [t,T]$

$$\begin{aligned}
&\mathbb{E}\left[|Y^1_{\gamma_t}(s) - Y^2_{\gamma_t}(s)|^2\right] + \frac{1}{2}\mathbb{E}\left[\int_s^T \|Z^1_{\gamma_t}(r) - Z^2_{\gamma_t}(r)\|^2 dr\right] + \frac{1}{2}\mathbb{E}\left[\int_s^T \int_{\mathbb{R}^l} |U^1_{\gamma_t}(r,x) - U^2_{\gamma_t}(r,x)|^2 \nu(dr,dx)\right] \\
&\le 2K\mathbb{E}\left[\int_s^T |Y^1_{\gamma_t}(r-) - Y^2_{\gamma_t}(r-)|^2 dr\right]. \qquad (3.14)
\end{aligned}$$

In particular, Gronwall's lemma yields $0 \le \mathbb{E}\left[|Y^1_{\gamma_t}(s) - Y^2_{\gamma_t}(s)|^2\right] \le 0$. Therefore, it is with inequality (3.14)

$$0 \le \frac{1}{2}\mathbb{E}\left[\int_s^T \|Z^1_{\gamma_t}(r) - Z^2_{\gamma_t}(r)\|^2 dr\right] \le 0 \quad \text{and} \quad 0 \le \frac{1}{2}\mathbb{E}\left[\int_s^T \int_{\mathbb{R}^l} |U^1_{\gamma_t}(r,x) - U^2_{\gamma_t}(r,x)|^2 \nu(dr,dx)\right] \le 0.$$

Moreover, Lemma 3.1 and the Lipschitz condition of the generator f yield

$$\mathbb{E}\left[\sup_{s\in[t,T]} |Y^1_{\gamma_t}(s) - Y^2_{\gamma_t}(s)|^2\right] \le K\left(\mathbb{E}\left[\int_t^T \|Z^1_{\gamma_t}(r) - Z^2_{\gamma_t}(r)\|^2 dr\right] + \mathbb{E}\left[\int_t^T \int_{\mathbb{R}^l} |U^1_{\gamma_t}(r,x) - U^2_{\gamma_t}(r,x)|^2 \nu(dr,dx)\right]\right) \le 0$$

and the solution $(Y_{\gamma_t}, Z_{\gamma_t}, U_{\gamma_t}) \in \mathbb{H}^2[t,T]$ to the BSDE (3.13) is unique.

We show existence of a solution to BSDE (3.13) by the iterative procedure: $Y_0(s) = Z_0(s) = U_0(s,x) = 0$ for $s \in [t,T]$ and $x \in \mathbb{R}^l$. We consider the recursive equation

$$Y_{n+1}(s) = \Phi(X_{\gamma_t,T}) + \int_s^T f(r, X_{\gamma_t,r-}, \bar{Y}(r-), Z_n(r), U_n(r))dr - \int_s^T Z_{n+1}(r)dW(r) - \int_s^T \int_{\mathbb{R}^l} U_{n+1}(r,x)\tilde{N}(dr,dx), \qquad (3.15)$$

for $s \in [t,T]$. The n-th equation is uniquely solved with the first step. It is only to show that the sequence converges. With inequality (3.7) and the Lipschitz condition of the generator f we obtain

$$\begin{aligned}
&\mathbb{E}\left[|Y_{n+1}(s) - Y_n(s)|^2\right] + \mathbb{E}\left[\int_s^T \|Z_{n+1}(r) - Z_n(r)\|^2 dr\right] + \mathbb{E}\left[\int_s^T \int_{\mathbb{R}^l} |U_{n+1}(r,x) - U_n(r,x)|^2 \nu(dr,dx)\right] \\
&\le 2K\mathbb{E}\left[\int_s^T |Y_{n+1}(r-) - Y_n(r-)|^2 dr\right] + \frac{1}{2}\mathbb{E}\left[\int_s^T \|Z_n(r) - Z_{n-1}(r)\|^2 dr\right] \\
&\quad + \frac{1}{2}\mathbb{E}\left[\int_s^T \int_{\mathbb{R}^l} |U_n(r,x) - U_{n-1}(r,x)|^2 \nu(dr,dx)\right].
\end{aligned}$$

Furthermore, we define for $n \ge 0$: $y_{n+1}(s) = \mathbb{E}\left[\int_s^T |Y_{n+1}(r-) - Y_n(r-)|^2 dr\right]$, $z_{n+1}(s) = \mathbb{E}\left[\int_s^T \|Z_{n+1}(r) - Z_n(r)\|^2 dr\right]$ and $u_{n+1}(s) = \mathbb{E}\left[\int_s^T \int_{\mathbb{R}^l} |U_{n+1}(r,x) - U_n(r,x)|^2 \nu(dr,dx)\right]$. Therefore, the last inequality implies

$$-\frac{d}{ds}(y_{n+1}(s)e^{2Ks}) + e^{2Ks}z_{n+1}(s) + e^{2Ks}u_{n+1}(s) \le \frac{1}{2}\left\{e^{2Ks}z_n(s) + e^{2Ks}u_n(s)\right\}.$$

The convergence follows like in Pardoux and Peng (1990). Integration from s to T yields

$$y_{n+1}(s)+\int_s^T e^{2K(r-s)}z_{n+1}(r)dr+\int_s^T e^{2K(r-s)}u_{n+1}(r)dr\le\frac{1}{2}\int_s^T e^{2K(r-s)}z_n(r)dr+\frac{1}{2}\int_s^T e^{2K(r-s)}u_n(r)dr \tag{3.16}$$

and it follows

$$\int_t^T e^{2K(r-t)}z_{n+1}(r)dr+\int_t^T e^{2K(r-t)}u_{n+1}(r)dr\le\frac{1}{2}\int_t^T e^{2K(r-t)}z_n(r)dr+\frac{1}{2}\int_t^T e^{2K(r-t)}u_n(r)dr\le 2^{-n}\tilde{c}e^{2K(T-t)},$$

with $\tilde{c}=\int_t^T[z_1(r)+u_1(r)]dr$. Furthermore, Lemma 3.1 and the Lipschitz condition of the generator f yield

$$\mathbb{E}\left[\sup_{s\in[t,T]}|Y_{n+1}(s)-Y_n(s)|^2\right]\le CK\left\{\mathbb{E}\left[\int_t^T\|Z_n(r)-Z_{n-1}(r)\|^2dr\right]+\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^l}|U_n(r,x)-U_{n-1}(r,x)|^2\nu(dr,dx)\right]\right\},$$

from which it immediately follows that $\{Y_n\}\in S^2[t,T]$, $\{Z_n\}\in M^2[t,T]$ and $\{U_n\}\in L^2_N(t,T)$ are Cauchy sequences. Passing to the limit in (3.15) as $n\to\infty$, we obtain that $(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t})\in\mathbb{H}^2[t,T]$, defined by $Y_{\gamma_t}=\lim_{n\to\infty}Y_n$, $Z_{\gamma_t}=\lim_{n\to\infty}Z_n$ and $U_{\gamma_t}=\lim_{n\to\infty}U_n$, solves equation (3.13).

Third step: Now we want to solve BSDE (2.1). We define the iterative procedure $Y_0(s)=0$ and

$$Y_{n+1}(s)=\Phi(X_{\gamma_t,T})+\int_s^T f(r,X_{\gamma_t,r-},Y_n(r-),Z_{n+1}(r),U_{n+1}(r))dr-\int_s^T Z_{n+1}dW(r)-\int_s^T\int_{\mathbb{R}^l}U_{n+1}(r,x)\tilde{N}(dr,dx), \tag{3.17}$$

for $s\in[t,T]$. By the second step there exist a unique solution to the n-th equation. Convergence of this approximation: It is with the help of equation (3.7) and the Lipschitz condition of the generator f

$$\begin{aligned}&\mathbb{E}\left[|Y_{n+1}(s)-Y_n(s)|^2\right]+\frac{1}{2}\mathbb{E}\left[\int_s^T\|Z_{n+1}(r)-Z_n(r)\|^2dr\right]+\frac{1}{2}\mathbb{E}\left[\int_s^T\int_{\mathbb{R}^l}|U_{n+1}(r,x)-U_n(r,x)|^2\nu(dr,dx)\right]\\&\le\tilde{K}\left\{\mathbb{E}\left[\int_s^T|Y_{n+1}(r-)-Y_n(r-)|^2dr\right]+\mathbb{E}\left[\int_s^T|Y_n(r-)-Y_{n-1}(r-)|^2dr\right]\right\},\end{aligned}$$

for $\tilde{K}=\max\{2K,\frac{1}{2}\}$. Define $y_{n+1}(s)=\mathbb{E}[\int_s^T|Y_{n+1}(r-)-Y_n(r-)|^2dr]$ and we get

$$-\frac{d}{ds}y_{n+1}(s)-\tilde{K}y_{n+1}(s)\le\tilde{K}y_n(s),\quad y_{n+1}(T)=0.$$

Similar to the second step it is

$$-\frac{d}{ds}\big(y_{n+1}(s)e^{\tilde{K}s}\big)\le\tilde{K}y_n(s)e^{\tilde{K}s}$$

and we obtain

$$y_{n+1}(s)\le\tilde{K}\int_s^T e^{\tilde{K}(r-s)}y_n(r)dr\le\frac{(\tilde{K}e^{\tilde{K}(T-t)})^n}{n!}y_1(t).$$

This implies that $\{Z_n\}$ is a Cauchy sequence in $M^2[t,T]$ and $\{U_n\}$ is a Cauchy sequence in $L^2_N(t,T)$. Moreover, it is with Lemma 3.1 and the Lipschitz condition of the generator f

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[t,T]}|Y_{n+1}(s)-Y_n(s)|^2\right]\le CK\Bigg\{&\mathbb{E}\left[\int_t^T|Y_n(r-)-Y_{n-1}(r-)|^2dr\right]+\mathbb{E}\left[\int_t^T\|Z_{n+1}(r)-Z_n(r)\|^2dr\right]\\&+\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^l}|U_{n+1}(r,x)-U_n(r,x)|^2\nu(dr,dx)\right]\Bigg\}.\end{aligned} \tag{3.18}$$

Now, $\{Y_n\}$ is a Cauchy sequence in $S^2[t,T]$ and $(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t})\in\mathbb{H}^2[t,T]$, where $Y_{\gamma_t}=\lim_{n\to\infty}Y_n$, $Z_{\gamma_t}=\lim_{n\to\infty}Z_n$ and $U_{\gamma_t}=\lim_{n\to\infty}U_n$ solves BSDE (2.1).

Now, we only have to prove the estimate (3.11): Applying Itô's formula to $|Y_{\gamma_t}(s)|^2$ we get for $s \in [t,T]$

$$\begin{aligned}
&|Y_{\gamma_t}(s)|^2 + \int_s^T \|Z_{\gamma_t}(r)\|^2 dr + \int_s^T \int_{\mathbb{R}^l} |U_{\gamma_t}(r,x)|^2 \nu(dr,dx) \\
&= |\Phi(X_T^{\gamma_t})|^2 + 2\int_s^T \langle Y_{\gamma_t}(r-), f(r, X_{\gamma_t,r-}, Y_{\gamma_t}(r-), Z_{\gamma_t}(r), U_{\gamma_t}(r))\rangle dr - 2\int_s^T \langle Y_{\gamma_t}(r-), Z_{\gamma_t}(r)\rangle dW(r) \\
&\quad - 2\int_s^T \int_{\mathbb{R}^l} \langle Y_{\gamma_t}(r-), U_{\gamma_t}(r,x)\rangle \tilde{N}(dr,dx) - \int_s^T \int_{\mathbb{R}^l} |U_{\gamma_t}(r,x)|^2 \tilde{N}(dr,dx)
\end{aligned}$$

and similar to the argumentation in the proof of Lemma 3.1 it is with the Lipschitz property of the generator f and inequality (3.5)

$$\begin{aligned}
&\mathbb{E}\left[|Y_{\gamma_t}(s)|^2\right] + \mathbb{E}\left[\int_s^T \|Z_{\gamma_t}(r)\|^2 dr\right] + \mathbb{E}\left[\int_s^T \int_{\mathbb{R}^l} |U_{\gamma_t}(r,x)|^2 \nu(dr,dx)\right] \\
&\leq c\left\{\mathbb{E}\left[|\Phi(X_{\gamma_t,T})|^2\right] + \left(\alpha + \frac{2K}{\alpha}\right)\mathbb{E}\left[\int_s^T |Y_{\gamma_t}(r-)|^2 dr\right] + \frac{2}{\alpha}\mathbb{E}\left[\int_s^T |f_0(r, X_{\gamma_t,r-})|^2 dr\right]\right\},
\end{aligned} \tag{3.19}$$

for $\alpha > 2K$. In particular, Gronwall's inequality yields

$$\mathbb{E}\left[|Y_{\gamma_t}(s)|^2\right] \leq c\left(\mathbb{E}\left[|\Phi(X_{\gamma_t,T})|^2\right] + \frac{2}{\alpha}\mathbb{E}\left[\int_s^T |f_0(r, X_{\gamma_t,r-})|^2 dr\right]\right) e^{\left(\alpha + \frac{2K}{\alpha}\right)(T-s)} \tag{3.20}$$

and we get with the last inequality for $s = t$

$$\mathbb{E}\left[\int_t^T \|Z_{\gamma_t}(r)\|^2 dr\right] + \mathbb{E}\left[\int_t^T \int_{\mathbb{R}^l} |U_{\gamma_t}(r,x)|^2 \nu(dr,dx)\right] \leq \tilde{C}\mathbb{E}\left[|\Phi(X_{\gamma_t,T})|^2 + \int_t^T |f_0(r, X_{\gamma_t,r-})|^2 dr\right]. \tag{3.21}$$

With Lemma 3.1 and the Lipschitz condition we obtain with the help of the inequalities (3.21) and (3.20)

$$\mathbb{E}\left[\sup_{s\in[t,T]} |Y_{\gamma_t}(s)|^2\right] \leq \tilde{C}\mathbb{E}\left[|\Phi(X_{\gamma_t,T})|^2 + \int_s^T |f_0(r, X_{\gamma_t,r-})|^2 dr\right].$$

Together with (3.21) this provides the estimate (3.11). ☐

Corollary 3.4. *Consider the BSDE with a generator f in which the last component u enters as an integral with respect to Q,*

$$\begin{aligned}
Y_{\gamma_t}(s) = \Phi(X_{\gamma_t,T}) &+ \int_s^T \left[m\left(r, X_{\gamma_t,r-}, Y_{\gamma_t}(r-), Z_{\gamma_t}(r), \int_{\mathbb{R}^l} U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right) + A(r)\right] dr \\
&- \int_s^T Z_{\gamma_t}(r) dW(r) - \int_s^T \int_{\mathbb{R}^l} U_{\gamma_t}(r,x)\tilde{N}(dr,dx), \qquad s \in [t,T],
\end{aligned} \tag{3.22}$$

where $\Phi \in L^2\big(D([0,T],\mathbb{R}^d);\mathbb{R}^m\big)$ and $\delta : \Omega \times [t,T] \times \mathbb{R}^l \to \mathbb{R}^m$ is predictable, non-negative and the mapping $s \to \int_{\mathbb{R}^l} |\delta(s,x)|^2 Q(s,dx)\eta(s)$ is bounded. Furthermore, let $m : \Omega \times [t,T] \times D([0,T],\mathbb{R}^d) \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m \to \mathbb{R}^m$ be predictable and linear in (y,z,u) such that $m(s,\gamma,0,0,0) = 0$ a.s. for all $\gamma \in D([0,T],\mathbb{R}^d)$, $s \in [t,T]$, and such that

$$|m(\omega,s,\gamma,y,z,u)| \leq K(s)(|y| + \|z\| + |u|),$$

where $K := \sup_{s\in[t,T]} |K(s)|^2 < \infty$. Moreover, let A be predictable with $\mathbb{E}[\int_t^T |A(r)|^2 dr] < \infty$. Then for all $\gamma \in D([0,T],\mathbb{R}^d)$ this BSDE admits a unique adapted solution $(Y_{\gamma_t}, Z_{\gamma_t}, U_{\gamma_t}) \in \mathbb{H}^2[t,T]$ and the following estimate holds

$$\|(Y_{\gamma_t}, Z_{\gamma_t}, U_{\gamma_t})\|^2_{\mathbb{H}^2[t,T]} \leq C\mathbb{E}\left[|\Phi(X_{\gamma_t,T})|^2 + \int_t^T |A(r)|^2 dr\right]. \tag{3.23}$$

Moreover, we assume that (Φ, f) and $(\bar{\Phi}, \bar{f})$ satisfy (A1)-(A3), for any $\gamma, \bar{\gamma} \in D([0,T],\mathbb{R}^d)$ and $t, \bar{t} \in [0,T]$. Let $(Y_{\gamma_t}, Z_{\gamma_t}, U_{\gamma_t}) \in \mathbb{H}^2[t,T]$, $(\bar{Y}_{\bar{\gamma}_{\bar{t}}}, \bar{Z}_{\bar{\gamma}_{\bar{t}}}, \bar{U}_{\bar{\gamma}_{\bar{t}}}) \in \mathbb{H}^2[\bar{t},T]$ be solutions to this BSDE with (Φ, f) and $(\bar{\Phi}, \bar{f})$. Then we have the estimate

$$\|(Y_{\gamma_t}, Z_{\gamma_t}, U_{\gamma_t}) - (\bar{Y}_{\bar{\gamma}_{\bar{t}}}, \bar{Z}_{\bar{\gamma}_{\bar{t}}}, \bar{U}_{\bar{\gamma}_{\bar{t}}})\|^2_{\mathbb{H}^2[t\vee\bar{t},T]} \leq C\Big(\mathbb{E}\Big[|\Phi(X_{\gamma_t,T}) - \bar{\Phi}(X_{\bar{\gamma}_{\bar{t}},T})|^2\Big] + \mathbb{E}\Big[\int_{t\vee\bar{t}}^T |A(r) - \bar{A}(r)|^2 dr\Big]\Big). \tag{3.24}$$

Proof. The first part of this corollary is a consequence of Theorem 3.3, since the assumptions of the corollary are stronger than the assumptions (A1)-(A3). The second part follows from Lemma 3.1 with the assumptions on m. ☐

4 Comparison Theorem

Now, let $m=1$ which implies that the processes Y_{γ_t} and U_{γ_t} are real-valued and the process Z_{γ_t} is $\mathbb{R}^{1\times d}$-valued. Consider the path-dependent BSDE (2.1) with the generator $f:\Omega\times[t,T]\times D([0,T],\mathbb{R}^d)\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathscr{M}(\mathbb{R}^l;\mathbb{R})\to\mathbb{R}$ and the terminal $\Phi:\Omega\times D([0,T],\mathbb{R}^d)\to\mathbb{R}$.

Theorem 4.1. *Let (f,Φ) and $(\bar f,\bar\Phi)$ be the parameters of BSDE (2.1). Assume that $(\bar f,\bar\Phi)$ satisfies (A1)-(A3). For (f,Φ) we assume (A1) and (A3) and we replace assumption (A2) by*

(A'2) *The generator $f:\Omega\times[t,T]\times D([0,T],\mathbb{R}^d)\times\mathbb{R}\times\mathbb{R}^d\times\mathscr{M}(\mathbb{R}^l;\mathbb{R})\to\mathbb{R}$ is predictable and*

$$|f(\omega,s,\gamma,y,z,u)-f(\omega,s,\bar\gamma,\bar y,\bar z,u)|\le K(s)\left[\|\gamma-\bar\gamma\|_D+|y-\bar y|+|z-\bar z|\right],$$
$$f(\omega,s,\gamma,y,z,u)-f(\omega,s,\gamma,y,z,\bar u)\le\int_{\mathbb{R}^l}\delta^{\gamma,y,z,u,\bar u}(s,x)\big(u(x)-\bar u(x)\big)Q(\omega,s,dx)\eta(\omega,s)$$

a.e. $(\omega,s)\in\Omega\times[t,T]$ for all $(\gamma,y,z,u),(\bar\gamma,\bar y,\bar z,u),(\gamma,y,z,\bar u)\in D([0,T],\mathbb{R}^d)\times\mathbb{R}\times\mathbb{R}^d\times L^2_{Q(\omega)}(\mathbb{R}^l;\mathbb{R})$ and $K:=\sup_{s\in[t,T]}|K(s)|^2<\infty$. The process $\delta^{\gamma,y,z,u,\bar u}:\Omega\times[t,T]\times\mathbb{R}^l\to(-1,\infty)$ is predictable such that the mapping $s\mapsto\int_{\mathbb{R}^l}|\delta^{\gamma,y,z,u,\bar u}(s,x)|^2Q(\omega,s,dx)\eta(\omega,s)$ is uniformly bounded in $(\gamma,y,z,u,\bar u)$.

Let $(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t})$ and $(\bar Y_{\gamma_t},\bar Z_{\gamma_t},\bar U_{\gamma_t})$ be the associated unique solutions in $\mathbb{H}^2[t,T]$. We suppose that

- *$\Phi(\gamma)\ge\bar\Phi(\gamma)$ a.e. $\gamma\in D([0,T],\mathbb{R}^d)$,*
- *$f(s,\gamma,y,z,u)\ge\bar f(s,\gamma,y,z,u)$, for $(s,\gamma,y,z,u)\in[t,T]\times D([0,T],\mathbb{R}^d)\times\mathbb{R}\times\mathbb{R}^d\times L^2_{(\omega)}(\mathbb{R}^l;\mathbb{R})$,*

then $Y_{\gamma_t}(s)\ge\bar Y_{\gamma_t}(s)$ for $s\in[t,T]$. In addition, if $Y_{\gamma_t}(s_0)=\bar Y_{\gamma_t}(s_0)$ a.s. for some $s_0\in[t,T]$, then $Y(s)=\bar Y(s)$ for $s\in[s_0,T]$.

Proof. In this proof we will use similar techniques as in Delong (2013). From assumption (A'2) we deduce that the generator f satisfies

$$f(\omega,r,\gamma,y,z,u)-f(\omega,r,\gamma,y,z,\bar u)\ge\int_{\mathbb{R}^l}\tilde\delta^{\gamma,y,z,u,\bar u}(r,x)\big(u(x)-\bar u(x)\big)Q(\omega,r,dx)\eta(\omega,r)\tag{4.1}$$

a.e. $(\omega,r)\in\Omega\times[t,T]$ for all $(\gamma,y,z,u),(\gamma,y,z,\bar u)\in D([0,T],\mathbb{R}^d)\times\mathbb{R}\times\mathbb{R}^d\times L^2_{Q(\omega)}(\mathbb{R}^l;\mathbb{R})$, where the process $\tilde\delta^{\gamma,y,z,u,\bar u}:\Omega\times[t,T]\times\mathbb{R}^l\to(-1,\infty)$ is predictable such that the mapping $s\mapsto\int_{\mathbb{R}^l}|\tilde\delta^{\gamma,y,z,u,\bar u}(s,x)|^2Q(s,dx)\eta(s)$ is uniformly bounded in $(\gamma,y,z,u,\bar u)$. Together with (A'2) the generator is Lipschitz continuous in γ, y, z and u in the sense of (A2) and Theorem 3.3 yields that there exist a unique solution $(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t})\in\mathbb{H}^2[t,T]$ to the BSDE (2.1) with (Φ,f) and for a given $\gamma\in D([0,T],\mathbb{R}^d)$, $t\in[0,T]$. Moreover, Theorem 3.3 yields that the BSDE with $(\bar\Phi,\bar f)$ has a unique solution $(\bar Y_{\gamma_t},\bar Z_{\gamma_t},\bar U_{\gamma_t})\in\mathbb{H}^2[t,T]$ for a given $\gamma\in D([0,T],\mathbb{R}^d)$, $t\in[0,T]$ fixed. For

$$\hat\Phi(X_{\gamma_t,T})=\Phi(X_{\gamma_t,T})-\bar\Phi(X_{\gamma_t,T})\in\mathbb{R},\qquad \hat Y_{\gamma_t}(s)=Y_{\gamma_t}(s)-\bar Y_{\gamma_t}(s)\in\mathbb{R},$$
$$\hat Z_{\gamma_t}(s)=Z_{\gamma_t}(s)-\bar Z_{\gamma_t}(s)\in\mathbb{R}^d,\qquad \hat f(s,\gamma,y,z,u)=f(s,\gamma,y,z,u)-\bar f(s,\gamma,y,z,u)\in\mathbb{R},$$

we get

$$\begin{aligned}\hat Y_{\gamma_t}(s)=\hat\Phi(X_{\gamma_t,T})&+\int_s^T f(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),U_{\gamma_t}(r))-\bar f(r,X_{\gamma_t,r-},\bar Y_{\gamma_t}(r-),\bar Z_{\gamma_t}(r),\bar U_{\gamma_t}(r))dr\\&-\int_s^T\hat Z_{\gamma_t}(r)dW(r)-\int_s^T\int_{\mathbb{R}^l}\hat U_{\gamma_t}(r,x)\tilde N(dr,dx).\end{aligned}$$

Applying Itô's formula for semimartingales to $g(s,x)=e^{\int_t^s\Delta_yf(u)du}x$ yields

$$\begin{aligned}\hat Y_{\gamma_t}(s)&=e^{\int_s^T\Delta_yf(u)du}\hat\Phi(X_{\gamma_t,T})+\int_s^T e^{\int_s^r\Delta_yf(u)du}\Delta_zf(r)|\hat Z_{\gamma_t}(r)|^2dr\\&+\int_s^T e^{\int_s^r\Delta_yf(u)du}\big(f(r,X_{\gamma_t,r-},\bar Y_{\gamma_t}(r-),\bar Z_{\gamma_t}(r),U_{\gamma_t}(r))-f(r,X_{\gamma_t,r-},\bar Y_{\gamma_t}(r-),\bar Z_{\gamma_t}(r),\bar U_{\gamma_t}(r))\big)dr\\&+\int_s^T e^{\int_s^r\Delta_yf(u)du}\hat f\big(r,X_{\gamma_t,r-},\bar Y_{\gamma_t}(r-),\bar Z_{\gamma_t}(r),\bar U_{\gamma_t}(r)\big)dr-\int_s^T e^{\int_s^r\Delta_yf(u)du}\hat Z_{\gamma_t}(r)dW(r)\\&-\int_s^T\int_{\mathbb{R}^l}e^{\int_s^r\Delta_yf(u)du}\big(U_{\gamma_t}(r,x)-\bar U_{\gamma_t}(r,x)\big)\tilde N(dr,dx),\end{aligned}\tag{4.2}$$

where

$$\Delta_y f(s) = \frac{f(s,X_{\gamma_t,s-},Y_{\gamma_t}(s-),Z_{\gamma_t}(s),U_{\gamma_t}(s)) - f(s,X_{\gamma_t,s-},\bar{Y}_{\gamma_t}(s-),Z_{\gamma_t}(s),U_{\gamma_t}(s))}{Y_{\gamma_t}(s-)-\bar{Y}_{\gamma_t}(s-)} \cdot \mathbb{1}_{\{\hat{Y}_{\gamma_t}(s-)\neq 0\}},$$

$$\Delta_z f(s) = \frac{f(s,X_{\gamma_t,s-},\bar{Y}_{\gamma_t}(s-),Z_{\gamma_t}(s),U_{\gamma_t}(s)) - f(s,X_{\gamma_t,s-},\bar{Y}_{\gamma_t}(s-),\bar{Z}_{\gamma_t}(s),U_{\gamma_t}(s))}{|Z_{\gamma_t}(s)-\bar{Z}_{\gamma_t}(s)|^2} \cdot \mathbb{1}_{\{\hat{Z}_{\gamma_t}(s)\neq 0\}}.$$

We remark that the processes $\Delta_y f$ and $\Delta_z f$ are bounded by condition (A'2). With inequality (4.1) we get

$$\begin{aligned}\hat{Y}_{\gamma_t}(s) \geq{}& e^{\int_s^T \Delta_y f(u)du}\hat{\Phi}(X_{\gamma_t,T}) + \int_s^T e^{\int_s^r \Delta_y f(u)du}\Delta_z f(r)|\hat{Z}_{\gamma_t}(r)|^2 dr \\ &+ \int_s^T \int_{\mathbb{R}^l} e^{\int_s^r \Delta_y f(u)du}\tilde{\delta}(r,x)\big(U_{\gamma_t}(r,x)-\bar{U}_{\gamma_t}(r,x)\big)\nu(dr,dx) \\ &+ \int_s^T e^{\int_s^r \Delta_y f(u)du}\hat{f}\big(r,X_{\gamma_t,r-},\bar{Y}_{\gamma_t}(r-),\bar{Z}_{\gamma_t}(r),\bar{U}_{\gamma_t}(r)\big)dr - \int_s^T e^{\int_s^r \Delta_y f(u)du}\hat{Z}_{\gamma_t}(r)dW(r) \\ &- \int_s^T \int_{\mathbb{R}^l} e^{\int_s^r \Delta_y f(u)du}\big(U_{\gamma_t}(r,x)-\bar{U}_{\gamma_t}(r,x)\big)\tilde{N}(dr,dx). \end{aligned} \tag{4.3}$$

Furthermore, we define an equivalent probability measure $\mathbb{Q}$ by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{F}_s^t} = M(s), \qquad \frac{dM(s)}{M(s-)} = \hat{Z}_{\gamma_t}(s)\Delta_z f(s)dW(s) + \int_{\mathbb{R}^l}\tilde{\delta}(s,x)\tilde{N}(ds,dx) \quad s\in[t,T]. \tag{4.4}$$

Since $\hat{Z}_{\gamma_t} \in M^2[t,T]$ and the assumption that f is predictable, the process $\hat{Z}_{\gamma_t}(s)\Delta_z f(s)$ is predictable. Furthermore, it is $|\hat{Z}_{\gamma_t}(s)\Delta_z f(s)| \leq K$. Since $\tilde{\delta}^{\gamma,y,z,u,\bar{u}}: \Omega\times[t,T]\times\mathbb{R}^l \to (-1,\infty)$ and the mapping $s\mapsto \int_{\mathbb{R}^l}|\tilde{\delta}^{\gamma,y,z,u,\bar{u}}(s,x)|^2 Q(s,dx)\eta(s)$ is uniformly bounded in $(\gamma,y,z,u,\bar{u})$ we get $\tilde{\delta}(s,x) > -1$, for $s\in[t,T]$, $x\in\mathbb{R}^l$, and $\int_{\mathbb{R}^l}|\tilde{\delta}(s,x)|^2 Q(s,dx)\eta(s) \leq K$ for $s\in[t,T]$. Together, Proposition 2.5.1 in Delong (2013) yields that the process M defined by (4.4) is a square integrable positive martingale and this implies that M is a uniformly integrable martingale. Hence, M defines an equivalent probability measure, see Protter (2004) Definition III.8.1. Applying the Girsanov-Meyer Theorem, see Theorem III8.35 in Protter (2004), we obtain $W^{\mathbb{Q}}(s) = W(s) - \int_t^s \hat{Z}_{\gamma_t}(r)\Delta_z f(r)dr$, for $s\in[t,T]$, where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-local martingale. Theorem III8.42 in Protter (2004) yields that $W^{\mathbb{Q}}$ is a standard Brownian motion. Theorem 2.5.1 in Delong (2013) provides that $\tilde{N}^{\mathbb{Q}}(s,A) = N(s,A) - \int_t^s\int_{\mathbb{R}^l}\big(1+\tilde{\delta}(r,x)\big)\nu(dr,dx)$ is a $(\mathbb{Q},\mathscr{F})$-compensated random measure. With (4.3) we derive

$$\begin{aligned}\hat{Y}_{\gamma_t}(s) \geq{}& e^{\int_s^T \Delta_y f(u)du}\hat{\Phi}(X_{\gamma_t,T}) + \int_s^T e^{\int_s^r \Delta_y f(u)du}\hat{f}\big(r,X_{\gamma_t,r-},\bar{Y}_{\gamma_t}(r-),\bar{Z}_{\gamma_t}(r),\bar{U}_{\gamma_t}(r)\big)dr - \int_s^T e^{\int_s^r \Delta_y f(u)du}\hat{Z}_{\gamma_t}(r)dW^{\mathbb{Q}}(r) \\ &- \int_s^T \int_{\mathbb{R}^l} e^{\int_s^r \Delta_y f(u)du}\big(U_{\gamma_t}(r,x)-\bar{U}_{\gamma_t}(r,x)\big)\tilde{N}^{\mathbb{Q}}(dr,dx), \end{aligned} \tag{4.5}$$

where the stochastic integrals are local martingales under $\mathbb{Q}$. Applying the Burkholder-Davis-Gundy inequality, the boundedness of $\Delta_y f$ and the Cauchy-Schwarz inequality we get

$$\mathbb{E}^{\mathbb{Q}}\left[\sup_{s\in[t,T]}\Big|\int_t^s e^{\int_s^r \Delta_y f(u)du}\hat{Z}_{\gamma_t}(r)dW^{\mathbb{Q}}(r)\Big|\right] \leq K\mathbb{E}\left[\Big|\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|^2\right]^{\frac{1}{2}}\mathbb{E}\left[\int_t^T |\hat{Z}_{\gamma_t}(r)|^2 dr\right]^{\frac{1}{2}} < \infty.$$

Similar it is

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}\left[\sup_{s\in[t,T]}\Big|\int_t^s \int_{\mathbb{R}^l} e^{\int_s^r \Delta_y f(u)du}\big(U_{\gamma_t}(r,x)-\bar{U}_{\gamma_t}(r,x)\big)\tilde{N}^{\mathbb{Q}}(dr,dx)\Big|\right] \\ &\leq C\mathbb{E}\left[\Big|\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|^2\right]^{\frac{1}{2}}\mathbb{E}\left[\int_t^T\int_{\mathbb{R}^l}|U_{\gamma_t}(r,x)-\bar{U}_{\gamma_t}(r,x)|^2\nu(dr,dx)\right]^{\frac{1}{2}} < \infty.\end{aligned}$$

With Theorem I.51 in Protter (2004) the two local martingales are $\mathbb{Q}$-martingales. Taking the expected value of (4.5) under the measure $\mathbb{Q}$ we deduce for $\hat{Y}_{\gamma_t} \in S^2[t,T]$

$$\hat{Y}_{\gamma_t}(s) \geq \mathbb{E}^{\mathbb{Q}}\left[e^{\int_s^T \Delta_y f(u)du}\hat{\Phi}(X_{\gamma_t,T}) + \int_s^T e^{\int_s^r \Delta_y f(u)du}\hat{f}\big(r,X_{\gamma_t,r-},\bar{Y}_{\gamma_t}(r-),\bar{Z}_{\gamma_t}(r),\bar{U}_{\gamma_t}(r)\big)dr\Big|\mathscr{F}_s^t\right] \geq 0, \tag{4.6}$$

with the assumptions on the generator f and the terminals Φ and $\bar{\Phi}$.

Moreover, for $s_0 \in [t,T]$ it follows

$$\hat{Y}_{\gamma_t}(s_0) \geq \mathbb{E}^{\mathbb{Q}}\left[e^{\int_{s_0}^T \Delta_y f(u)du}\hat{\Phi}(X_{\gamma_t,T}) + \int_{s_0}^T e^{\int_{s_0}^r \Delta_y f(u)du}\hat{f}\big(r,X_{\gamma_t,r-},\bar{Y}_{\gamma_t}(r-),\bar{Z}_{\gamma_t}(r),\bar{U}_{\gamma_t}(r)\big)dr\Big|\mathscr{F}^t_{s_0}\right] \geq 0.$$

If $Y_{\gamma_t}(s_0) = \bar{Y}_{\gamma_t}(s_0)$ for some $s_0 \in [t,T]$ we get with the assumptions on the generator and the terminal that $\Phi(X_{\gamma_t,T}) = \bar{\Phi}(X_{\gamma_t,T})$ a.s. and $f(s,X_{\gamma_t,s-},\bar{Y}_{\gamma_t}(s-),\bar{Z}_{\gamma_t}(s),\bar{U}_{\gamma_t}(s)) = \bar{f}(s,X_{\gamma_t,s-},\bar{Y}_{\gamma_t}(s-),\bar{Z}_{\gamma_t}(s),\bar{U}_{\gamma_t}(s))$ a.s. for almost every $s \in [s_0,T]$. Furthermore, equation (4.2) provides

$$\begin{aligned}\hat{Y}_{\gamma_t}(s) = &\int_s^T e^{\int_s^r \Delta_y f(u)du}\Delta_z f(r)|\hat{Z}_{\gamma_t}(r)|^2 dr\\ &+\int_s^T e^{\int_s^r \Delta_y f(u)du}\big(f(r,X_{\gamma_t,r-},\bar{Y}_{\gamma_t}(r-),\bar{Z}_{\gamma_t}(r),U_{\gamma_t}(r)) - f(r,X_{\gamma_t,r-},\bar{Y}_{\gamma_t}(r-),\bar{Z}_{\gamma_t}(r),\bar{U}_{\gamma_t}(r))\big)dr\\ &-\int_s^T e^{\int_s^r \Delta_y f(u)du}\hat{Z}_{\gamma_t}(r)dW(r) - \int_s^T\int_{\mathbb{R}^l} e^{\int_s^r \Delta_y f(u)du}\big(U_{\gamma_t}(r,x)-\bar{U}_{\gamma_t}(r,x)\big)\tilde{N}(dr,dx), \quad s\in[s_0,T].\end{aligned}$$

With assumption (A'2) it is

$$\begin{aligned}\hat{Y}_{\gamma_t}(s) \leq &\int_s^T e^{\int_s^r \Delta_y f(u)du}\Delta_z f(r)|\hat{Z}_{\gamma_t}(r)|^2 dr + \int_s^T\int_{\mathbb{R}^l} e^{\int_s^r \Delta_y f(u)du}\delta(r,x)\big(U_{\gamma_t}(r,x)-\bar{U}_{\gamma_t}(r,x)\big)\nu(dr,dx)\\ &-\int_s^T e^{\int_s^r \Delta_y f(u)du}\hat{Z}_{\gamma_t}(r)dW(r) - \int_s^T\int_{\mathbb{R}^l} e^{\int_s^r \Delta_y f(u)du}\big(U_{\gamma_t}(r,x)-\bar{U}_{\gamma_t}(r,x)\big)\tilde{N}(dr,dx).\end{aligned}$$

By change of measure with

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{F}^t_s} = M(s), \qquad \frac{dM(s)}{M(s-)} = \hat{Z}_{\gamma_t}(s)\Delta_z f(s)dW(s) + \int_{\mathbb{R}^l}\delta(s,x)\tilde{N}(ds,dx), \quad s\in[s_0,T]$$

we get for $s\in[s_0,T]$

$$\hat{Y}_{\gamma_t}(s) = -\int_s^T e^{\int_s^r \Delta_y f(u)du}\hat{Z}_{\gamma_t}(r)dW^{\mathbb{Q}}(r) - \int_s^T\int_{\mathbb{R}^l} e^{\int_s^r \Delta_y f(u)du}\big(U_{\gamma_t}(r,x)-\bar{U}_{\gamma_t}(r,x)\big)\tilde{N}^{\mathbb{Q}}(dr,dx).$$

Taking the conditional expectation under the measure $\mathbb{Q}$ yields $\hat{Y}_{\gamma_t}(s) \leq 0$ a.s. for a.e. $s\in[s_0,T]$. Since $\hat{Y}_{\gamma_t}(s)\geq 0$ we get $\hat{Y}_{\gamma_t}(s) = 0$ for all $s\in[s_0,T]$ and the last assertion is proved. □

Corollary 4.2. *Consider the following BSDE*

$$\begin{aligned}Y_{\gamma_t}(s) = \Phi(X_{\gamma_t,T}) &+ \int_s^T f\left(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),\int_{\mathbb{R}^l}U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right)dr\\ &-\int_s^T Z_{\gamma_t}(r)dW(r) - \int_s^T\int_{\mathbb{R}^l}U_{\gamma_t}(r,x)\tilde{N}(dr,dx), \qquad s\in[t,T]. \qquad (4.7)\end{aligned}$$

We assume

- $\Phi \in L^2\big(D([0,T],\mathbb{R}^d);\mathbb{R}\big)$,
- *the generator* $f:\Omega\times[t,T]\times D([0,T],\mathbb{R}^d)\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ *is predictable and Lipschitz continuous in the sense that*

$$|f(\omega,s,\gamma,y,z,u) - f(\omega,s,\bar{\gamma},\bar{y},\bar{z},\bar{u})|^2 \leq K(s)\Big[\|\gamma-\bar{\gamma}\|_D^2 + |y-\bar{y}|^2 + |z-\bar{z}|^2 + |u-\bar{u}|^2\Big], \qquad (4.8)$$

  a.e. $(\omega,s)\in\Omega\times[t,T]$ *for all* $(\gamma,y,z,u),(\bar{\gamma},\bar{y},\bar{z},\bar{u})\in D([0,T],\mathbb{R}^d)\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$*, and with* $\sup_{s\in[t,T]}|K(s)|^2<\infty$,
- $\mathbb{E}[\int_t^T|f_0(r,\gamma_{r-})|^2dr]<\infty$*, with* $f_0(r,\gamma_r) := f(r,\gamma_r,0,0,0)$,
- *the mapping* $u\mapsto f(s,\gamma,y,z,u)$ *is non-decreasing for each* $(s,\gamma,y,z)\in[t,T]\times D([0,T],\mathbb{R}^d)\times\mathbb{R}\times\mathbb{R}^d$,

- *the process $\delta : \Omega \times [t,T] \times \mathbb{R}^l \to \mathbb{R}$ is predictable, non-negative and the mapping $s \mapsto \int_{\mathbb{R}^l} |\delta(s,x)|^2 Q(s,dx)\eta(s)$ is bounded.*

Let $(\bar{\Phi}, \bar{f})$ satisfy the same assumptions. Let $(Y_{\gamma_t}, Z_{\gamma_t}, U_{\gamma_t}), (\bar{Y}_{\gamma_t}, \bar{Z}_{\gamma_t}, \bar{U}_{\gamma_t}) \in \mathbb{H}^2[t,T]$ be the unique solutions to this BSDE for a fixed $\gamma \in D([0,T],\mathbb{R}^d)$ and $t \in [0,T]$ with (Φ, f) and $(\bar{\Phi}, \bar{f})$. If

- *$\Phi(\gamma) \geq \bar{\Phi}(\gamma)$ a.e. $\gamma \in D([0,T],\mathbb{R}^d)$,*
- *$f(s,\gamma,y,z,u) \geq \bar{f}(s,\gamma,y,z,u)$, for $(s,\gamma,y,z,u) \in [t,T] \times D([0,T],\mathbb{R}^d) \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$,*

then $Y_{\gamma_t}(s) \geq \bar{Y}_{\gamma_t}(s)$ for $s \in [t,T]$. Furthermore, if $Y_{\gamma_t}(s_0) = \bar{Y}_{\gamma_t}(s_0)$ a.s. for some $s_0 \in [t,T]$, then $Y_{\gamma_t}(s) = \bar{Y}_{\gamma_t}(s)$ for $s \in [s_0,T]$.

Proof. Theorem 3.3 yields the existence and uniqueness of the solutions $(Y_{\gamma_t}, Z_{\gamma_t}, U_{\gamma_t}), (\bar{Y}_{\gamma_t}, \bar{Z}_{\gamma_t}, \bar{U}_{\gamma_t})$. Similar to the argumentation in the proof of Theorem 4.1 it is for

$$\hat{\Phi}(X_{\gamma_t,T}) = \Phi(X_{\gamma_t,T}) - \bar{\Phi}(X_{\gamma_t,T}) \in \mathbb{R}, \qquad \hat{Y}_{\gamma_t}(s) = Y_{\gamma_t}(s) - \bar{Y}_{\gamma_t}(s) \in \mathbb{R},$$
$$\hat{Z}_{\gamma_t}(s) = Z_{\gamma_t}(s) - \bar{Z}_{\gamma_t}(s) \in \mathbb{R}^d, \qquad \hat{U}_{\gamma_t}(s,x) = U_{\gamma_t}(s,x) - \bar{U}_{\gamma_t}(s,x) \in \mathbb{R},$$

and $\hat{f}(s,\gamma,y,z,u) = f(s,\gamma,y,z,u) - \bar{f}(s,\gamma,y,z,u) \in \mathbb{R}$, with Itô's formula

$$\begin{aligned}
\hat{Y}_{\gamma_t}(s) = {} & \hat{\Phi}(X_{\gamma_t,T}) e^{\int_s^T \Delta_y f(u)du} + \int_s^T e^{\int_s^r \Delta_y f(u)du} \Delta_z f(r) |\hat{Z}_{\gamma_t}(r)|^2 dr \\
& + \int_s^T \int_{\mathbb{R}^l} e^{\int_s^r \Delta_y f(u)du} \Delta_u f(r) \hat{U}_{\gamma_t}(r,x) \delta(r,x) Q(r,dx) \eta(r) dr \\
& + \int_s^T e^{\int_s^r \Delta_y f(u)du} \hat{f}\left(r, X_{\gamma_t,r-}, \bar{Y}_{\gamma_t}(r-), \bar{Z}_{\gamma_t}(r), \int_{\mathbb{R}^l} \bar{U}_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right) dr - \int_s^T e^{\int_s^r \Delta_y f(u)du} \hat{Z}_{\gamma_t}(r) dW(r) \\
& - \int_s^T \int_{\mathbb{R}^l} e^{\int_s^r \Delta_y f(u)du} \hat{U}_{\gamma_t}(r,x) \tilde{N}(dr,dx),
\end{aligned}$$

for

$$\begin{aligned}
\Delta_y f(s) = {} & \Bigg(f\left(s, X_{\gamma_t,s-}, Y_{\gamma_t}(s-), Z_{\gamma_t}(s), \int_{\mathbb{R}^l} U_{\gamma_t}(s,x)\delta(s,x)Q(s,dx)\eta(s)\right) \\
& \quad - f\left(s, X_{\gamma_t,s-}, \bar{Y}_{\gamma_t}(s-), Z_{\gamma_t}(s), \int_{\mathbb{R}^l} U_{\gamma_t}(s,x)\delta(s,x)Q(s,dx)\eta(s)\right)\Bigg) \Big/ \left(Y_{\gamma_t}(s-) - \bar{Y}_{\gamma_t}(s-)\right) \cdot \mathbb{1}_{\{\hat{Y}_{\gamma_t}(s-)\neq 0\}}, \\
\Delta_z f(s) = {} & \Bigg(f\left(s, X_{\gamma_t,s-}, \bar{Y}_{\gamma_t}(s-), Z_{\gamma_t}(s), \int_{\mathbb{R}^l} U_{\gamma_t}(s,x)\delta(s,x)Q(s,dx)\eta(s)\right) \\
& \quad - f\left(s, X_{\gamma_t,s-}, \bar{Y}_{\gamma_t}(s-), \bar{Z}_{\gamma_t}(s), \int_{\mathbb{R}^l} U_{\gamma_t}(s,x)\delta(s,x)Q(s,dx)\eta(s)\right)\Bigg) \Big/ |Z_{\gamma_t}(s) - \bar{Z}_{\gamma_t}(s)|^2 \cdot \mathbb{1}_{\{\hat{Z}_{\gamma_t}(s)\neq 0\}}, \\
\Delta_u f(s) = {} & \Bigg(f\left(s, X_{\gamma_t,s-}, \bar{Y}_{\gamma_t}(s-), \bar{Z}_{\gamma_t}(s), \int_{\mathbb{R}^l} U_{\gamma_t}(s,x)\delta(s,x)Q(s,dx)\eta(s)\right) \\
& \quad - f\left(s, X_{\gamma_t,s-}, \bar{Y}_{\gamma_t}(s-), \bar{Z}_{\gamma_t}(s), \int_{\mathbb{R}^l} \bar{U}_{\gamma_t}(s,x)\delta(s,x)Q(s,dx)\eta(s)\right)\Bigg) \\
& \quad \Big/ \left(\int_{\mathbb{R}^l} (U_{\gamma_t}(s,x) - \bar{U}_{\gamma_t}(s,x))\delta(s,x)Q(s,dx)\eta(s)\right) \cdot \mathbb{1}_{\{\hat{U}_{\gamma_t}(s)\neq 0\}}.
\end{aligned}$$

Furthermore, we get an equivalent probability measure $\mathbb{Q}$ by the Radon-Nikodym derivative

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathscr{F}_s^t} = M(s), \qquad \frac{dM(s)}{M(s-)} = \Delta_z f(s) \hat{Z}_{\gamma_t}(s) dW(s) + \int_{\mathbb{R}^l} \Delta_u f(s) \delta(s,x) \tilde{N}(ds,dx), \qquad s \in [t,T].$$

The assumptions yield that the process M is a square integrable martingale and thus, M defines an equivalent probability measure. Arguing as in the proof of Theorem 4.1 we get

$$\hat{Y}_{\gamma_t}(s) = \hat{\Phi}(X_{\gamma_t,T})e^{\int_s^T \Delta_y f(u)du} + \int_s^T e^{\int_s^r \Delta_y f(u)du}\hat{f}\left(r, X_{\gamma_t,r-}, \bar{Y}_{\gamma_t}(r-), \bar{Z}_{\gamma_t}(r), \int_{\mathbb{R}^l} \bar{U}_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right)dr$$
$$- \int_s^T e^{\int_s^r \Delta_y f(u)du}\hat{Z}_{\gamma_t}(r)dW^{\mathbb{Q}}(r) - \int_s^T \int_{\mathbb{R}^l} e^{\int_s^r \Delta_y f(u)du}\hat{U}_{\gamma_t}(r,x)\tilde{N}^{\mathbb{Q}}(dr,dx), \qquad s \in [t,T].$$

Similar to the proof of Theorem 4.1 we conclude that the stochastic integrals in the equation above are true $\mathbb{Q}$-martingales and we get the first assertion by taking the conditional expectation. The assumptions of the second assertion yield $\hat{Y}_{\gamma_t}(s_0) = 0$, $\hat{\Phi}(X_{\gamma_t,T}) \geq 0$ and $\hat{f}(r, X_{\gamma_t,r-}, \bar{Y}_{\gamma_t}(r-), \bar{Z}_{\gamma_t}(r), \int_{\mathbb{R}^l} \bar{U}_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)) \geq 0$. By taking the conditional expectation we deduce that $\hat{\Phi}(X_{\gamma_t,T}) = 0$ a.s. and

$$\hat{f}(r, X_{\gamma_t,r-}, \bar{Y}_{\gamma_t}(r-), \bar{Z}_{\gamma_t}(r), \int_{\mathbb{R}^l} \bar{U}_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)) = 0$$

a.s. a.e. $s \in [s_0,T]$. This implies

$$\hat{Y}_{\gamma_t}(s) = -\int_s^T e^{\int_s^r \Delta_y f(u)du}\hat{Z}_{\gamma_t}(r)dW^{\mathbb{Q}}(r) - \int_s^T \int_{\mathbb{R}^l} e^{\int_s^r \Delta_y f(u)du}\hat{U}_{\gamma_t}(r,x)\tilde{N}^{\mathbb{Q}}(dr,dx), \qquad s \in [t,T],$$

and the second assertion follows by taking the conditional expectation. □

5 Path-differentiability of BSDEs

In this section we study the path-differentiability of BSDEs of type (4.7). We assume

(B1) $\Phi \in L^2(D([0,T],\mathbb{R}^d);\mathbb{R}^m)$ is continuous, $D_i^1\Phi(\gamma,[t,T])$ and $D_{ij}^2\Phi(\gamma,[t,T])$ exist and are continuous. Furthermore, let $D_i^1\Phi$, $D_{ij}^2\Phi \in L^2(D([0,T],\mathbb{R}^d);\mathbb{R}^m)$, $i,j = 1,\ldots,d$,

(B2) the generator $f : \Omega \times [t,T] \times D([0,T],\mathbb{R}^d) \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m \to \mathbb{R}^m$ is predictable and Lipschitz continuous, i.e.

$$|f(\omega,s,\gamma,y,z,u) - f(\omega,s,\bar{\gamma},\bar{y},\bar{z},\bar{u})|^2$$
$$\leq K(s)\left(\|\gamma-\bar{\gamma}\|_D^2 + |y-\bar{y}|^2 + \|z-\bar{z}\|^2 + |u-\bar{u}|^2\right),$$

a.e. $(\omega,s) \in \Omega\times[t,T]$ for all $(\gamma,y,z,u),(\bar{\gamma},\bar{y},\bar{z},\bar{u}) \in D([0,T],\mathbb{R}^d)\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m$, $K := \sup_{s\in[t,T]}|K(s)|^2 < \infty$,

(B3) $\mathbb{E}[\int_t^T |f_0(r,\gamma_{r-})|^2 dr] < \infty$,

(B4) for all $\tilde{\gamma} \in D([0,T],\mathbb{R}^d)$ and $s \in [t,T]$, $(x,s,y,z,u) \mapsto f(s,\tilde{\gamma}_s + x\mathbb{1}_{[t,T]},y,z,u)$ and

$$(x,s,y,z,u) \mapsto D_i^1 f(s,\tilde{\gamma}_s + x\mathbb{1}_{[t,T]},y,z,u;[t,T]) + f_y(s,\tilde{\gamma}_s + x\mathbb{1}_{[t,T]},y,z,u)D_i^1 y(s-;[t,T])$$
$$+ f_z(s,\tilde{\gamma}_s + x\mathbb{1}_{[t,T]},y,z,u)D_i^1 z(s;[t,T])$$
$$+ \int_{\mathbb{R}^l} f_u(s,\tilde{\gamma}_s + x\mathbb{1}_{[t,T]},y,z,u)D_i^1 u(s,x;[t,T])Q(r,dx)\eta(s),\ i = 1,\ldots,d,$$

are of class $C^1(\mathbb{R}^d\times[t,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m,\mathbb{R}^m)$ and the first order partial derivatives with respect to (x,y,z,u) are uniformly bounded in $\tilde{\gamma}$.

(B5) $\delta : \Omega\times[t,T]\times\mathbb{R}^l \to \mathbb{R}^m$ is predictable, non-negative and for $s \in [t,T]$ the mapping $s \to \int_{\mathbb{R}^l}|\delta(s,x)|^2 Q(s,dx)\eta(s)$ is bounded,

(B6) let $\gamma \mapsto U_{\gamma_t}(s,x)$ be of class $C^2(D([0,T],\mathbb{R}^d);\mathbb{R}^m)$ for any $(s,x) \in [t,T]\times\mathbb{R}^l$ and assume that

$$\int_{\mathbb{R}^l} |\varphi(\gamma_t,s,x)|Q(r,dx)\eta(r) < \infty,$$

where $\varphi(\gamma_t,s,x)=D_i^1U_{\gamma_t}(r,x;[t,T]),D_{ij}^2U_{\gamma_t}(r,x;[t,T])$, $i,j=1,\dots,d$. Furthermore, assume that there exists a neighborhood $(-\varepsilon,\varepsilon)$ and processes M_{γ_t} and N_{γ_t} such that for each $s\in[t,T]$

$$\int_{\mathbb{R}^l}|M_{\gamma_t}(r,x)|Q(r,dx)\eta(r)<\infty,\quad \int_{\mathbb{R}^l}|N_{\gamma_t}(r,x)|Q(r,dx)\eta(r)<\infty$$

and such that for $i,j=1,...,d$

$$\frac{|U_{\gamma_t+h\mathbb{1}_{[t,T]}e_i}(r,x)-U_{\gamma_t}(r,x)|}{h}\le M^i_{\gamma_t}(r,x),\qquad h\in(-\varepsilon,\varepsilon)$$

and

$$\frac{|D_j^1U_{\gamma_t+h\mathbb{1}_{[t,T]}e_i}(r,x)-D_j^1U_{\gamma_t}(r,x)|}{h}\le N^j_{\gamma_t}(r,x),\qquad h\in(-\varepsilon,\varepsilon)$$

Remark 5.1. *With assumption (B6) it follows easily*

$$D_i^1\int_{\mathbb{R}^l}U_{\gamma_t}(r,x)Q(r,dx)\eta(r)=\int_{\mathbb{R}^l}D_i^1U_{\gamma_t}(r,x)Q(r,dx)\eta(r) \tag{5.1}$$

for $i=1,\dots,d$ and for each $r\in[t,T]$.

Theorem 5.2. *Let (B1)-B(6) hold. Then the function*

$$D([0,T],\mathbb{R}^d)\to\mathbb{H}^2[t,T],\qquad \gamma_t\mapsto(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t})$$

is differentiable in the sense of Definition 2.2 where $\mathbb{R}^m$ is replaced by the Banach space $\mathbb{H}^2[t,T]$. The derivative is the unique solution to BDSE

$$\begin{aligned}
&D_i^1Y_{\gamma_t}(s;[t,T])\\
&=D_i^1\Phi(X_{\gamma_t,T};[t,T])+\int_s^T D_i^1f\left(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),\int_{\mathbb{R}^l}U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r);[t,T]\right)dr\\
&\quad+\int_s^T\Bigg[\langle f_y\left(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),\int_{\mathbb{R}^l}U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right),D_i^1Y_{\gamma_t}(r-;[t;T])\rangle\\
&\quad+\langle f_z\left(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),\int_{\mathbb{R}^l}U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right),D_i^1Z_{\gamma_t}(r;[t,T])\rangle\Bigg]dr\\
&\quad+\int_s^T\int_{\mathbb{R}^l}\langle f_u\left(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),\int_{\mathbb{R}^l}U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right),D_i^1U_{\gamma_t}(r,x;[t,T])\rangle\nu(dr,dx)\\
&\quad-\int_s^T D_i^1Z_{\gamma_t}(r;[t,T])dW(r)-\int_s^T\int_{\mathbb{R}^l}D_i^1U_{\gamma_t}(r,x;[t,T])\tilde N(dr,dx),\qquad i=1,\dots,d,\quad s\in[t,T].
\end{aligned}\tag{5.2}$$

Proof. To simplify the notation we restrict ourselves to the one-dimensional case, $m=d=l=1$. The extension to the multidimensional case is straightforward. We define

$$\begin{aligned}
&\mathscr{Y}^h(s)=\frac{1}{h}\left(Y_{\gamma_t^h}(s)-Y_{\gamma_t}(s)\right),\quad \mathscr{Z}^h(s)=\frac{1}{h}\left(Z_{\gamma_t^h}(s)-Z_{\gamma_t}(s)\right),\quad \mathscr{U}^h(s,x)=\frac{1}{h}\left(U_{\gamma_t^h}(s,x)-U_{\gamma_t}(s,x)\right),\\
&\Delta\Phi^h(X_{\gamma_t,T})=\frac{1}{h}\left(\Phi(X_{\gamma_t^h,T})-\Phi(X_{\gamma_t,T})\right),
\end{aligned}$$

for $s\in[t,T]$ and $x\in\mathbb{R}$. With the definition of X^{γ_t} we get $X_{\gamma_t^h,T}(u)-X_{\gamma_t,T}(u)=h\mathbb{1}_{[t,T]}(u)$, $u\in[0,T]$. It follows for $h\in\mathbb{R}\setminus\{0\}$ that $(\mathscr{Y}^h,\mathscr{Z}^h,\mathscr{U}^h)$ is the solution to BSDE

$$\begin{aligned}
\mathscr{Y}^h(s)&=\frac{1}{h}\left(\Phi(X_{\gamma_t^h,T})-\Phi(X_{\gamma_t,T})\right)+\frac{1}{h}\int_s^T\Bigg[f\left(r,X_{\gamma_t^h,r-},Y_{\gamma_t^h}(r-),Z_{\gamma_t^h}(r),\int_{\mathbb{R}}U_{\gamma_t^h}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right)\\
&\quad-f\left(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),\int_{\mathbb{R}}U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right)\Bigg]dr\\
&\quad-\int_s^T\mathscr{Z}^h(r)dW(r)-\int_s^T\int_{\mathbb{R}}\mathscr{U}^h(r,x)\tilde N(dr,dx),\qquad s\in[t,T].
\end{aligned}\tag{5.3}$$

Define for any $s \in [t,T]$, $h \in \mathbb{R} \setminus \{0\}$, the mapping $l_h^s : [0,1] \to D([0,T],\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by

$$l_h^s(\theta)$$
$$= \Big(X_{\gamma_t,s} + \theta(X_{\gamma_t^h,s} - X_{\gamma_t,s}), Y_{\gamma_t}(s-) + \theta h \mathscr{Y}^h(s-), Z_{\gamma_t}(s) + \theta h \mathscr{Z}^h(s), \int_{\mathbb{R}} \big(U_{\gamma_t}(s,x) + \theta h \mathscr{U}^h(s,x) \big) \delta(s,x) Q(s,dx) \eta(s) \Big)$$

and it follows for $s \in [t,T]$

$$\frac{1}{h} \frac{\partial}{\partial \theta} l_h^s(\theta) = \left(1, \mathscr{Y}^h(s-), \mathscr{Z}^h(s), \int_{\mathbb{R}} \mathscr{U}^h(s,x) \delta(s,x) Q(s,dx) \eta(s) \right).$$

With this we obtain

$$\begin{aligned}
&\frac{1}{h} f\left(s, X_{\gamma_t^h,s-}, Y_{\gamma_t^h}(s-), Z_{\gamma_t^h}(s), \int_{\mathbb{R}} U_{\gamma_t^{he}}(s,x) \delta(s,x) Q(s,dx) \eta(s) \right) \\
&- f\left(s, X_{\gamma_t,s-}, Y_{\gamma_t}(s-), Z_{\gamma_t}(s), \int_{\mathbb{R}} U_{\gamma_t}(s,x) \delta(s,x) Q(s,dx) \eta(s) \right) \\
&= \frac{1}{h} \int_0^1 \frac{\partial}{\partial \theta} f\big(l_h^s(\theta) \big) d\theta \\
&= \int_0^1 \langle \nabla f\big(l_h^s(\theta) \big), \frac{1}{h} \frac{\partial}{\partial \theta} l_h^s(\theta) \rangle d\theta \\
&= A_h^1(s) + A_h^2(s) \mathscr{Y}^h(s-) + A_h^3(s) \mathscr{Z}^h(s) + A_h^4(s) \int_{\mathbb{R}} \mathscr{U}^h(s,x) \delta(s,x) Q(s,dx) \eta(s),
\end{aligned}$$

for

$$A_h^1(s) = \int_0^1 D^1 f(l_h^s(\theta); [t,T]) d\theta, \qquad A_h^2(s) = \int_0^1 \frac{\partial}{\partial y} f(l_h^s(\theta)) d\theta,$$
$$A_h^3(s) = \int_0^1 \frac{\partial}{\partial z} f(l_h^s(\theta)) d\theta, \qquad A_h^4(s) = \int_0^1 \frac{\partial}{\partial u} f(l_h^s(\theta)) d\theta.$$

Furthermore, we define a linear function $m_h^s : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $m_h^s(y,z,u) = A_h^2(s) y + A_h^3(s) z + A_h^4(s) u$. This provides that $(\mathscr{Y}^h, \mathscr{Z}^h, \mathscr{U}^h)$ is the solution to linearized BSDE

$$\begin{aligned}
\mathscr{Y}^h(s) = \frac{1}{h} \left(\Phi(X_{\gamma_t^h,T}) - \Phi(X_{\gamma_t,T}) \right) + \int_s^T \left[m_h^s \left(\mathscr{Y}^h(r-), \mathscr{Z}^h(r), \int_{\mathbb{R}} \mathscr{U}^h(r,x) \delta(r,x) Q(r,dx) \eta(r) \right) + A_h^1(r) \right] dr \\
- \int_s^T \mathscr{Z}^h(r) dW(r) - \int_s^T \int_{\mathbb{R}} \mathscr{U}^h(r,x) \tilde{N}(dr,dx), \qquad s \in [t,T]. \qquad (5.4)
\end{aligned}$$

Besides, from the assumptions the conditions of Corollary 3.4 are satisfied and we get with the inequality (3.24)

$$\mathbb{E}\left[\sup_{s \in [t,T]} |\mathscr{Y}^h(s) - \mathscr{Y}^{\bar{h}}(s)|^2 \right] \le C \left(\mathbb{E}\left[|\Delta\Phi^h(X_{\gamma_t,T}) - \Delta\Phi^{\bar{h}}(X_{\gamma_t,T})|^2 \right] + \mathbb{E}\left[\int_t^T |A_h^1(r) - A_{\bar{h}}^1(r)|^2 dr \right] \right). \qquad (5.5)$$

From (B1) it is

$$\lim_{h \to \bar{h}} \mathbb{E}\left[|\Delta\Phi^h(X_{\gamma_t,T}) - \Delta\Phi^{\bar{h}}(X_{\gamma_t,T})|^2 \right] = \lim_{h \to \bar{h}} \mathbb{E}\left[|\Phi(X_{\gamma_t^{\bar{h}},T}) - \Phi(X_{\gamma_t^h,T})|^2 \right] = 0$$

and (B4) provides

$$\lim_{h \to \bar{h}} \mathbb{E}\left[\int_t^T \left| A_h^1(r) - A_{\bar{h}}^1(r) \right|^2 dr \right] \le \lim_{h \to \bar{h}} \mathbb{E}\left[\int_t^T \int_0^1 \left| D^1 f(l_h^r(\theta); [t,T]) - D^1 f(l_{\bar{h}}^r(\theta); [t,T]) \right|^2 d\theta dr \right] = 0.$$

Thus, it is $\lim_{h \to \bar{h}} \mathbb{E}[\sup_{s \in [t,T]} |\mathscr{Y}^h(s) - \mathscr{Y}^{\bar{h}}(s)|^2] = 0$. With the same arguments we get

$$\lim_{h \to \bar{h}} \mathbb{E}\left[\int_t^T |\mathscr{Z}^h(s) - \mathscr{Z}^{\bar{h}}(s)|^2 ds \right] = 0 \quad \text{and} \quad \lim_{h \to \bar{h}} \mathbb{E}\left[\int_t^T \int_{\mathbb{R}} |\mathscr{U}^h(s,x) - \mathscr{U}^{\bar{h}}(s,x)|^2 \nu(ds,dx) \right] = 0.$$

Finally, this gives us

$$\lim_{h\to\bar{h}} \|(\mathscr{Y}^h, \mathscr{Z}^h, \mathscr{U}^h) - (\mathscr{Y}^{\bar{h}}, \mathscr{Z}^{\bar{h}}, \mathscr{U}^{\bar{h}})\|^2_{\mathbb{H}^2[t,T]} = 0.$$

Since $S^2[t,T]$, $M^2[t,T]$ and $L^2_N(t,T)$ are Banach spaces the sequence $\mathscr{Y}^{h_n}(s)$ converges to a process $D^1Y_{\gamma_t}(s;[t,T])$, the sequence $\mathscr{Z}^{h_n}(s)$ converges to a process $D^1Z_{\gamma_t}(s;[t,T])$ and the process $\mathscr{U}^{h_n}(s,x)$ converges to $D^1U_{\gamma_t}(s,x;[t,T])$ for a sequence $(h_n)_n \in \mathbb{R}\setminus\{0\}$ and with respect to the corresponding norms.

This implies that $(D^1Y_{\gamma_t}(\cdot;[t,T]), D^1Z_{\gamma_t}(\cdot;[t,T]), D^1U_{\gamma_t}(\cdot,\cdot;[t,T])) \in \mathbb{H}^2[t,T]$ is a solution to the BSDE

$$\begin{aligned}
&D^1Y_{\gamma_t}(s;[t,T]) \\
&= D^1\Phi(X_{\gamma_t,T};[t,T]) + \int_s^T D^1 f\left(r, X_r^{\gamma_t}, Y_{\gamma_t}(r-), Z_{\gamma_t}(r), \int_{\mathbb{R}} U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r);[t,T]\right) dr \\
&\quad + \int_s^T \Big[f_y\left(r, X_{\gamma_t,r-}, Y_{\gamma_t}(r-), Z_{\gamma_t}(r), \int_{\mathbb{R}} U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right) D^1Y_{\gamma_t}(r-;[t;T]) \\
&\quad + f_z\left(r, X_{\gamma_t,r-}, Y_{\gamma_t}(r-), Z_{\gamma_t}(r), \int_{\mathbb{R}} U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right) D^1Z_{\gamma_t}(r;[t,T])\Big] dr \\
&\quad + \int_s^T \int_{\mathbb{R}} f_u\left(r, X_{\gamma_t,r-}, Y_{\gamma_t}(r-), Z_{\gamma_t}(r), \int_{\mathbb{R}} U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right) D^1U_{\gamma_t}(r,x;[t,T])\nu(dr,dx) \\
&\quad - \int_s^T D^1Z_{\gamma_t}(r;[t,T])dW(r) - \int_s^T \int_{\mathbb{R}} D^1U_{\gamma_t}(r,x;[t,T])\tilde{N}(dr,dx), \qquad s\in[t,T]. \qquad (5.6)
\end{aligned}$$

This proves our statement. □

Corollary 5.3. *With assumptions (B1) - (B6) the function* $D([0,T],\mathbb{R}^d) \to \mathbb{H}^2[t,T]$ *with* $\gamma \mapsto (Y_{\gamma_t}, Z_{\gamma_t}, U_{\gamma_t})$ *is of class* $C^2(D([0,T],\mathbb{R}^d);\mathbb{R}^m)$.

Proof. We consider only the case $m = d = l = 1$ and define the process

$$V_{\gamma_t}(s) = \left(Y_{\gamma_t}(s-), Z_{\gamma_t}(s), \int_{\mathbb{R}} U_{\gamma_t}(s,x)\delta(s,x)Q(s,dx)\eta(s)\right)$$

to simplify the notation. Analogous to the proof of Theorem 5.2 we define for $\varepsilon \in \mathbb{R}\setminus\{0\}$

$$\begin{aligned}
\mathscr{Y}^\varepsilon(s) &= \frac{1}{\varepsilon}\left(D^1Y_{\gamma_t^\varepsilon}(s;[t,T]) - D^1Y_{\gamma_t}(s;[t,T])\right), \quad \mathscr{U}^\varepsilon(s,x) = \frac{1}{\varepsilon}\left(D^1U_{\gamma_t^\varepsilon}(s,x;[t,T]) - D^1U_{\gamma_t}(s,x;[t,T])\right), \\
\mathscr{Z}^\varepsilon(s) &= \frac{1}{\varepsilon}\left(D^1Z_{\gamma_t^\varepsilon}(s;[t,T]) - D^1Z_{\gamma_t}(s;[t,T])\right), \quad \mathscr{H}^\varepsilon(X_{\gamma_t,T}) = \frac{1}{\varepsilon}\left(D^1\Phi(X_{\gamma_t^\varepsilon,T};[t,T]) - D^1\Phi(X_{\gamma_t,T};[t,T])\right).
\end{aligned}$$

Furthermore we will use the notation

$$\begin{aligned}
&\mathscr{G}_f\left(r, X_{\gamma_t,r-}, V_{\gamma_t}(r), D^1Y_{\gamma_t}(r-;[t,T]), D^1Z_{\gamma_t}(r;[t,T]), \int_{\mathbb{R}} D^1U_{\gamma_t}(r,x;[t,T])\delta(r,x)Q(r,dx)\eta(r)\right) \\
&= D^1 f\left(r, X_{\gamma_t,r-}, V_{\gamma_t}(r);[t,T]\right) + f_y\left(r, X_{\gamma_t,r-}, V_{\gamma_t}(r)\right) D^1Y_{\gamma_t}(r-;[t;T]) \\
&\quad + f_z\left(r, X_{\gamma_t,r-}, V_{\gamma_t}(r)\right) D^1Z_{\gamma_t}(r;[t,T]) + \int_{\mathbb{R}} f_u\left(r, X_{\gamma_t,r-}, V_{\gamma_t}(r)\right) D^1U_{\gamma_t}(r,x;[t,T])Q(r,dx)\eta(r)
\end{aligned}$$

and define for any $s\in[t,T]$ the mapping $l^s_\varepsilon : [0,1] \to D([0,T],\mathbb{R})\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ by

$$\begin{aligned}
l^s_\varepsilon(\theta) = \Big(&X_{\gamma_t,s-} + \theta(X_{\gamma_t^\varepsilon,s-} - X_{\gamma_t,s-}), V_{\gamma_t}(s) + \theta(V_{\gamma_t^\varepsilon}(s) - V_{\gamma_t}(s)), D^1Y_{\gamma_t}(s-;[t,T]) + \theta\varepsilon\mathscr{Y}^\varepsilon(s-), \\
&D^1Z_{\gamma_t}(s;[t,T]) + \theta\varepsilon\mathscr{Z}^\varepsilon(s), \int_{\mathbb{R}} \left(D^1U_{\gamma_t}(s,x;[t,T]) + \theta\varepsilon\mathscr{U}^\varepsilon(s,x)\right)\delta(s,x)Q(s,dx)\eta(s)\Big).
\end{aligned}$$

With these definitions we immediately get

$$
\begin{aligned}
&\mathscr{Y}^{\varepsilon}(s) \\
&= \mathscr{H}^{\varepsilon}(X_{\gamma_t,T}) \\
&+ \frac{1}{\varepsilon}\int_s^T \Bigg[\mathscr{G}_f\left(r, X_{\gamma_t^\varepsilon,r-}, V_{\gamma_t^\varepsilon}(r), D^1Y_{\gamma_t^\varepsilon}(r-;[t,T]), D^1Z_{\gamma_t^\varepsilon}(r;[t,T]), \int_{\mathbb{R}} D^1U_{\gamma_t^\varepsilon}(r,x;[t,T])\delta(r,x)Q(r,dx)\eta(r)\right) \\
&\qquad - \mathscr{G}_f\left(r, X_{\gamma_t,r-}, V_{\gamma_t}(r), D^1Y_{\gamma_t}(r-;[t,T]), D^1Z_{\gamma_t}(r;[t,T]), \int_{\mathbb{R}} D^1U_{\gamma_t}(r,x;[t,T])\delta(r,x)Q(r,dx)\eta(r)\right)\Bigg]dr \\
&- \int_s^T \mathscr{Z}^{\varepsilon}(r)dW(r) - \int_s^T\int_{\mathbb{R}} \mathscr{U}^{\varepsilon}(r,x)\tilde{N}(dr,dx),
\end{aligned}
$$

where

$$
\begin{aligned}
&\frac{1}{\varepsilon}\Bigg[\mathscr{G}_f\left(s, X_{\gamma_t^\varepsilon,s-}, V_{\gamma_t^\varepsilon}(s), D^1Y_{\gamma_t^\varepsilon}(s-;[t,T]), D^1Z_{\gamma_t^\varepsilon}(s,[t,T]), \int_{\mathbb{R}} D^1U_{\gamma_t^\varepsilon}(s,x;[t,T])\delta(s,x)Q(s,dx)\eta(s)\right) \\
&\qquad - \mathscr{G}_f\left(s, X_{\gamma_t,s-}, V_{\gamma_t}(s), D^1Y_{\gamma_t}(s-;[t,T]), D^1Z_{\gamma_t}(s;[t,T]), \int_{\mathbb{R}} D^1U_{\gamma_t}(s,x;[t,T])\delta(s,x)Q(s,dx)\eta(s)\right)\Bigg] \\
&= \frac{1}{\varepsilon}\int_0^1 \frac{\partial}{\partial\theta}\mathscr{G}_f\big(l_\varepsilon^s(\theta)\big)d\theta \\
&= \int_0^1 \langle \nabla\mathscr{G}_f\big(l_\varepsilon^s(\theta)\big), \frac{1}{\varepsilon}\frac{\partial}{\partial\theta} l_\varepsilon^s(\theta)\rangle d\theta \\
&= A_\varepsilon^1(s) + \langle A_\varepsilon^2(s), \frac{1}{\varepsilon}\left(V_{\gamma_t^\varepsilon}(s) - V_{\gamma_t}(s)\right)\rangle + A_\varepsilon^3(s)\mathscr{Y}^{\varepsilon}(s-) + A_\varepsilon^4(s)\mathscr{Z}^{\varepsilon}(s) + A_\varepsilon^5(s)\int_{\mathbb{R}} \mathscr{U}^{\varepsilon}(s,x)\delta(s,x)Q(s,dx)\eta(s)
\end{aligned}
$$

and A_ε^i, $i = 1,\dots,4$ are given by

$$
\begin{aligned}
A_\varepsilon^1(s) &= \int_0^1 D^1\mathscr{G}_f(l_\varepsilon^s(\theta);[t,T])d\theta, \qquad A_\varepsilon^2(s) = \int_0^1 \frac{\partial}{\partial v}\mathscr{G}_f(l_\varepsilon^s(\theta))d\theta, \\
A_\varepsilon^3(s) &= \int_0^1 \frac{\partial}{\partial y'}\mathscr{G}_f(l_\varepsilon^s(\theta))d\theta, \qquad A_\varepsilon^4(s) = \int_0^1 \frac{\partial}{\partial z'}\mathscr{G}_f(l_\varepsilon^s(\theta))d\theta, \\
A_\varepsilon^5(s) &= \int_0^1 \frac{\partial}{\partial u'}\mathscr{G}_f(l_\varepsilon^s(\theta))d\theta.
\end{aligned}
$$

With our assumptions (B1) - (B6) and by using the same steps as in the proof of Theorem 5.2, we get

$$
\begin{aligned}
\lim_{\varepsilon\to\bar\varepsilon}\mathbb{E}[\sup_{s\in[t,T]}|\mathscr{Y}^{\varepsilon}(s) - \mathscr{Y}^{\bar\varepsilon}(s)|^2] &= 0, \\
\lim_{\varepsilon\to\bar\varepsilon}\mathbb{E}\left[\int_t^T |\mathscr{Z}^{\varepsilon}(s) - \mathscr{Z}^{\bar\varepsilon}(s)|^2 ds\right] &= 0, \\
\lim_{\varepsilon\to\bar\varepsilon}\mathbb{E}\left[\int_t^T\int_{\mathbb{R}} |\mathscr{U}^{\varepsilon}(s,x) - \mathscr{U}^{\bar\varepsilon}(s,x)|^2\nu(ds,dx)\right] &= 0,
\end{aligned}
$$

which gives us $\lim_{\varepsilon\to\bar\varepsilon}\|(\mathscr{Y}^{\varepsilon},\mathscr{Z}^{\varepsilon},\mathscr{U}^{\varepsilon}) - (\mathscr{Y}^{\bar\varepsilon},\mathscr{Z}^{\bar\varepsilon},\mathscr{U}^{\bar\varepsilon})\|^2_{\mathbb{H}^2[t,T]} = 0$. In our case, this implies that the sequence $\mathscr{Y}^{h_n}(s)$ converges to a process $D^2Y_{\gamma_t}(s;[t,T])$, the sequence $\mathscr{Z}^{h_n}(s)$ converges to a process $D^2Z_{\gamma_t}(s;[t,T])$ and the process $\mathscr{U}^{h_n}(s,x)$ converges to $D^2U_{\gamma_t}(s,x;[t,T])$ for a sequence $(h_n)_n \in \mathbb{R}\setminus\{0\}$ and with respect to the corresponding norms. □

Remark 5.4. *Corollary 5.3 implies that* $(D^2Y_{\gamma_t}(\cdot;[t,T]), D^2Z_{\gamma_t}(\cdot;[t,T]), D^2U_{\gamma_t}(\cdot,\cdot;[t,T])) \in \mathbb{H}^2[t,T]$ *is the unique so-*

lution to BSDE

$$
\begin{aligned}
&D^2Y_{\gamma_t}(s;[t,T])\\
&= D^2\Phi(X_{\gamma_t,T};[t,T]) + \int_s^T \Bigg[D^1\mathscr{G}_f\Bigg(r, X_{\gamma_t,r-}, V_{\gamma_t}(r), D^1Y_{\gamma_t}(r-;[t,T]), D^1Z_{\gamma_t}(r;[t,T]),\\
&\qquad \int_{\mathbb{R}} D^1U_{\gamma_t}(r,x;[t,T])\delta(r,x)Q(r,dx)\eta(r);[t,T]\Bigg) + \langle \frac{\partial}{\partial v}\mathscr{G}_f\Bigg(r, X_{\gamma_t,r-}, V_{\gamma_t}(r),\\
&\quad D^1Y_{\gamma_t}(r-;[t,T]), D^1Z_{\gamma_t}(r;[t,T]), \int_{\mathbb{R}} D^1U_{\gamma_t}(r,x;[t,T])\delta(r,x)Q(r,dx)\eta(r)\Bigg), D^1V_{\gamma_t}(r;[t;T])\rangle\Bigg]dr\\
&+ \int_s^T \Bigg[f_y\left(r, X_{\gamma_t,r-}, Y_{\gamma_t}(r-), Z_{\gamma_t}(r), \int_{\mathbb{R}} U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right) D^2Y_{\gamma_t}(r-;[t,T])\\
&+ f_z\left(r, X_{\gamma_t,r-}, Y_{\gamma_t}(r-), Z_{\gamma_t}(r), \int_{\mathbb{R}} U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right) D^2Z_{\gamma_t}(r;[t,T])\\
&+ \int_{\mathbb{R}} f_u\left(r, X_{\gamma_t,r-}, Y_{\gamma_t}(r-), Z_{\gamma_t}(r), \int_{\mathbb{R}} U_{\gamma_t}(r,x)\delta(r,x)Q(r,dx)\eta(r)\right) D^2U_{\gamma_t}(r,x;[t,T])\delta(r,x)Q(r,dx)\eta(r)\Bigg]dr\\
&- \int_s^T D^2Z_{\gamma_t}(r;[t,T])dW(r) - \int_s^T\int_{\mathbb{R}} D^2U_{\gamma_t}(r,x;[t,T])\tilde{N}(dr,dx), \qquad s\in[t,T]. \qquad (5.7)
\end{aligned}
$$

6 Viscosity solutions

For a given path $\gamma \in D([0,T],\mathbb{R}^d)$ we study the forward backward stochastic differential equation (FBSDE) consisting of a SDE of the form

$$
\begin{aligned}
&dX_{\gamma_t}(s) = b(s, X_{\gamma_t,s-})ds + \sigma(s, X_{\gamma_t,s-})dW(s) + \int_{\mathbb{R}^l} \beta(s, X_{\gamma_t,s-}, z)\tilde{N}(ds,dz), \quad s\in[t,T],\\
&X_{\gamma_t,t}(s) = \gamma_t(s), \quad s\in[0,t), \qquad (6.1)
\end{aligned}
$$

and a BSDE with a deterministic generator $f : [t,T]\times D([0,T],\mathbb{R}^d)\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{R}\to\mathbb{R}$,

$$
\begin{aligned}
dY_{\gamma_t}(s) &= -f\left(s, X_{\gamma_t,s-}, Y_{\gamma_t}(s-), Z_{\gamma_t}(s), \int_{\mathbb{R}^l} U_{\gamma_t}(s,z)\delta(s,z)\nu(dz)\right)ds + Z_{\gamma_t}(s)dW(s)\\
&\quad + \int_{\mathbb{R}^l} U_{\gamma_t}(s,z)\tilde{N}(ds,dz),\ s\in[t,T],\\
Y_{\gamma_t}(T) &= \Phi(X_{\gamma_t,T}), \qquad (6.2)
\end{aligned}
$$

where the random measure $N(ds,dz)$ is the integer-valued random measure and $\tilde{N}(ds,dz) = N(ds,dz) - ds\nu(dz)$ is the compensated Poisson measure with a Lévy measure ν. In other words we consider the case where $\eta(s) = 1$ and $Q(s,dz) = \nu(dz)$. Notice, that the path of the process X_{γ_t} defined in (6.1) satisfies equation (**??**).

Define $a(s,x_1,x_2) := \sigma(s,x_1)\sigma(s,x_2)^T$ for each $s\in[0,T]$, $x_1,x_2\in D([0,T],\mathbb{R}^d)$ and assume

(H0) σ and β are predictable processes with

$$\int_0^T |\sigma(s,x)|^2 ds < \infty, \quad \int_0^T\int_{\mathbb{R}^l} |\beta(s,x,z)|^2\nu(dz)ds < \infty,$$

(H1) for each $s\in[0,T]$ there exists $K_1(s) > 0$, with $K_1 := \sup_{s\in[0,T]} K_1(s) < \infty$, such that, for all $x_1,x_2\in D([0,T],\mathbb{R}^d)$,

$$
\begin{aligned}
&|b(s,x_1) - b(s,x_2)|^2 + \|a(s,x_1,x_1) - 2a(s,x_1,x_2) + a(s,x_2,x_2)\|\\
&+ \int_{\mathbb{R}^l} |\beta(s,x_1,z) - \beta(s,x_2,z)|^2\nu(dz) \le K_1(s)\sup_{u\in[0,s]} |x_1(u) - x_2(u)|^2,
\end{aligned}
$$

(H2) for each $s\in[0,T]$ there exists $K_2(s) > 0$, with $K_2 := \sup_{s\in[0,T]} K_2(s) < \infty$, such that, for all $x\in D([0,T],\mathbb{R}^d)$,

$$|b(s,x)|^2 + \|a(s,x,x)\| + \int_{\mathbb{R}^l} |\beta(s,x,z)|^2\nu(dz) \le K_2(s)(1 + \sup_{u\in[0,s]} |x(u)|^2),$$

(H3) the generator $f : [t,T] \times D([0,T],\mathbb{R}^d) \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition

$$|f(s,\gamma,y,z,u) - f(s,\bar{\gamma},\bar{y},\bar{z},\bar{u})| \le K(s)\left(\|\gamma-\bar{\gamma}\|_D + |y-\bar{y}| + \|z-\bar{z}\| + |u-\bar{u}|\right), \tag{6.3}$$

for all $(s,\gamma,y,z,u),(s,\bar{\gamma},\bar{y},\bar{z},\bar{u}) \in [t,T] \times D([0,T],\mathbb{R}^d) \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{R}$, $\sup_{s\in[t,T]} |K(s)| < \infty$,

(H4) the mapping $u \mapsto f(s,\gamma,y,z,u)$ is non-decreasing for each $(s,\gamma,y,z) \in [t,T] \times D([0,T],\mathbb{R}^d) \times \mathbb{R} \times \mathbb{R}^{1\times d}$,

(H5) the process $\delta : \Omega \times [t,T] \times \mathbb{R}^l \to \mathbb{R}$ is predictable, non-negative and the mapping

$$s \mapsto \int_{\mathbb{R}^l} |\delta(s,z)|^2 \nu(dz)$$

is bounded,

(H6) $\Phi : \Omega \times D([0,T],\mathbb{R}^d) \to \mathbb{R}$ with $|\Phi(\gamma) - \Phi(\bar{\gamma})| \le K\|\gamma-\bar{\gamma}\|_D$ and $|\Phi(\gamma)|^2 \le K(1+\|\gamma\|^2)$ for $\gamma,\bar{\gamma} \in D([0,T],\mathbb{R}^d)$,

(H7) $|f_0(s,\gamma)|^2 \le K(1+\|\gamma\|^2)$.

Under assumptions (H0) - (H2) Corollary 3.3 in Kromer et al. (2015) yields that for every $T > 0$ there exists a constant C depending only on K_2 and T such that for all $s \in [t,T]$

$$\mathbb{E}\left[\sup_{u\in[0,s]} |X_{\gamma_t,s}(u)|^2\right] = \mathbb{E}\left[\sup_{u\in[0,s]} |X_{\gamma_t}(u)|^2\right] \le C\left(1+\sup_{u\in[0,t]} |\gamma_t(u)|^2\right) e^{C(s-t)}. \tag{6.4}$$

In Keller (2016) the process X is a semimartingale and is given by its canonical representation in terms of its characteristics (B,C,ν), see Jacod and Shiryaev (2003). Furthermore, the definition of the path derivatives depends on conditions formulated in terms of probability measures.

Theorem 6.1. *Let $t \in [0,T]$ and $\gamma \in D([0,T],\mathbb{R}^d)$ with $\sup_{s\in[0,t]} |\gamma(s)|^2 < \infty$. Under the assumptions above there exists a unique solution $(X_{\gamma_t},Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t}) \in S^2[0,T] \times \mathbb{H}^2[t,T]$ to FBSDE (6.1), (6.2).*

Proof. This claim can be immediately deduced form Theorem 3.3 and Corollary 3.2 in Kromer et al. (2015). □

Corollary 6.2. *Under assumptions (H0) - (H7) there exists a constant C depending on t, T and K_2 such that*

$$\left\|\left(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t}\right)\right\|^2_{\mathbb{H}^2[t,T]} \le C\left(1+\sup_{s\in[0,t]} |\gamma_t(s)|^2\right) e^{C(T-t)}. \tag{6.5}$$

Proof. Theorem 3.3 provides

$$\left\|\left(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t}\right)\right\|^2_{\mathbb{H}^2[t,T]} \le C\left(\mathbb{E}\left[\left|\Phi(X_{\gamma_t,T})\right|^2\right] + \mathbb{E}\left[\int_t^T \left|f_0(r,X_{\gamma_t,r-})\right|^2 dr\right]\right)$$

and with assumptions (H6) - (H7) and the help of equation (6.4) it is

$$\left\|\left(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t}\right)\right\|^2_{\mathbb{H}^2[t,T]} \le C\left(1+\sup_{s\in[0,t]} |\gamma_t(s)|^2\right) e^{C(T-t)},$$

where C depends on t, T and K_2. □

Theorem 6.3. *Let assumptions (H0) - (H2) hold, $\gamma,\bar{\gamma} \in D([0,T],\mathbb{R}^d)$ and $t,\bar{t} \in [0,T]$ with $t \le \bar{t}$. If X_{γ_t} and $X_{\bar{\gamma}_{\bar{t}}}$ satisfy SDE (6.1) we get*

$$\mathbb{E}\left[\|X_{\gamma_t,T} - X_{\bar{\gamma}_{\bar{t}},T}\|^2_D\right] \le C\left(\sup_{s\in[0,\bar{t}]} |\gamma_t(s) - \bar{\gamma}_{\bar{t}}(s)|^2 + (\bar{t}-t)\left(1+\sup_{s\in[0,t]} |\gamma_t(s)|^2\right) e^{C(\bar{t}-t)}\right). \tag{6.6}$$

Proof. With equation (6.4) and assumption (H2) we get

$$\mathbb{E}\left[\int_t^{\bar{t}} \left|b\left(u, X_{\gamma_t,u-}\right)\right|^2 du\right] \leq \tilde{C}(\bar{t}-t)\left(1+\sup_{s\in[0,t]}|\gamma_t(s)|^2\right)e^{\tilde{C}(\bar{t}-t)}. \tag{6.7}$$

Since σ is a predictable process it is with Theorem 2.3.3. in Delong (2013) and assumption (H2)

$$\mathbb{E}\left[\left|\int_t^{\bar{t}} \sigma\left(u, X_{\gamma_t,u-}\right) dW(u)\right|^2\right] \leq \tilde{C}(\bar{t}-t)\left(1+\sup_{s\in[0,t]}|\gamma_t(s)|^2\right)e^{\tilde{C}(\bar{t}-t)}. \tag{6.8}$$

Similarly, we obtain

$$\mathbb{E}\left[\left|\int_t^{\bar{t}} \int_{\mathbb{R}^l} \beta\left(u, X_{\gamma_t,u-}, z\right) \tilde{N}(du,dz)\right|^2\right] \leq \tilde{C}(\bar{t}-t)\left(1+\sup_{s\in[0,t]}|\gamma_t(s)|^2\right)e^{\tilde{C}(\bar{t}-t)}. \tag{6.9}$$

Moreover, we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,\bar{t}]}\left|X_{\gamma_t,T}(s)-X_{\bar{\gamma}_{\bar{t}},T}(s)\right|^2\right] \\
&\leq \mathbb{E}\left[\sup_{s\in[0,\bar{t}]}|\gamma_t(s)-\bar{\gamma}_{\bar{t}}(s)|^2\right]+\mathbb{E}\left[\sup_{s\in[t,\bar{t}]}\left|X_{\gamma_t}(s)-\gamma_t(t)\right|^2\right] \\
&= \sup_{s\in[0,\bar{t}]}|\gamma_t(s)-\bar{\gamma}_{\bar{t}}(s)|^2+\mathbb{E}\left[\sup_{s\in[t,\bar{t}]}\left|\int_t^s b(r,X_{\gamma_t,r-})dr+\int_t^s \sigma(r,X_{\gamma_t,r-})dW(r)\right.\right. \\
&\quad \left.\left.+\int_t^s\int_{\mathbb{R}^l}\beta(r,X_{\gamma_t,r-},z)\tilde{N}(dr,dz)\right|^2\right].
\end{aligned}$$

With the Burkholder-Davis-Gundy inequality, Doob's martingale inequality and the equations above we obtain

$$\mathbb{E}\left[\sup_{s\in[0,\bar{t}]}\left|X_{\gamma_t,T}(s)-X_{\bar{\gamma}_{\bar{t}},T}(s)\right|^2\right] \leq \sup_{s\in[0,\bar{t}]}|\gamma_t(s)-\bar{\gamma}_{\bar{t}}(s)|^2+C(\bar{t}-t)\left(1+\sup_{s\in[0,t]}|\gamma_t(s)|^2\right)e^{C(\bar{t}-t)}. \tag{6.10}$$

Furthermore, it is with Jensen's inequality and assumption (H1)

$$\mathbb{E}\left[\sup_{s\in[\bar{t},T]}\left|\int_{\bar{t}}^s b(u,X_{\gamma_t,u-})-b(u,X_{\bar{\gamma}_{\bar{t}},u-})du\right|^2\right] \leq K_1\int_{\bar{t}}^T \mathbb{E}\left[\sup_{s\in[0,u]}\left|X_{\gamma_t,u-}(s)-X_{\bar{\gamma}_{\bar{t}},u-}(s)\right|^2\right]du.$$

Moreover, we obtain with the Burkholder-Davis-Gundy inequality

$$\mathbb{E}\left[\sup_{s\in[\bar{t},T]}\left|\int_{\bar{t}}^s \sigma(u,X_{\gamma_t,u-})-\sigma(u,X_{\bar{\gamma}_{\bar{t}},u-})dW(u)\right|^2\right] \leq \Lambda K_1\int_{\bar{t}}^T \mathbb{E}\left[\sup_{s\in[0,u]}\left|X_{\gamma_t,u-}(s)-X_{\bar{\gamma}_{\bar{t}},u-}(s)\right|^2\right]du.$$

Similarly, it follows

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[\bar{t},T]}\left|\int_{\bar{t}}^s\int_{\mathbb{R}^l} \beta(u,X_{\gamma_t,u-},z)-\beta(u,X_{\bar{\gamma}_{\bar{t}},u-},z)\tilde{N}(du,dz)\right|^2\right] \\
&\leq \Lambda K_1\int_{\bar{t}}^T \mathbb{E}\left[\sup_{\tau\in[0,u]}\left|X_{\gamma_t,u-}(\tau)-X_{\bar{\gamma}_{\bar{t}},u-}(\tau)\right|^2\right]du.
\end{aligned}$$

With equation (6.10) we get

$$
\begin{aligned}
&\int_{\bar t}^{T}\mathbb{E}\left[\sup_{s\in[0,u]}\left|X_{\gamma_t,u-}(s)-X_{\bar\gamma_t,u-}(s)\right|^2\right]du\\
&\le (T-\bar t)\left\{\sup_{s\in[0,\bar t]}|\gamma_t(s)-\bar\gamma_t(s)|^2+C(\bar t-t)\left(1+\sup_{s\in[0,t]}|\gamma_t(s)|^2\right)e^{C(\bar t-t)}\right\}\\
&\quad+\int_{\bar t}^{T}\mathbb{E}\left[\sup_{s\in[\bar t,u]}\left|X_{\gamma_t,u-}(s)-X_{\bar\gamma_t,u-}(s)\right|^2\right]du. \qquad (6.11)
\end{aligned}
$$

Since X_{γ_t} and $X_{\bar\gamma_t}$ satisfy SDE (6.1) it is

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[\bar t,T]}\left|X_{\gamma_t,T}(s)-X_{\bar\gamma_t,T}(s)\right|^2\right]\\
&\le 3\Bigg\{\sup_{s\in[0,\bar t]}|\gamma_t(s)-\bar\gamma_t(s)|^2+\mathbb{E}\Bigg[\sup_{s\in[\bar t,T]}\Bigg|\int_{\bar t}^{s}b(u,X_{\gamma_t,u-})-b(u,X_{\bar\gamma_t,u-})du\\
&\quad+\int_{\bar t}^{s}\sigma(u,X_{\gamma_t,u-})-\sigma(u,X_{\bar\gamma_t,u-})dW(u)+\int_t^s\int_{\mathbb{R}^l}\beta(u,X_{\gamma_t,u-},z)-\beta(u,X_{\bar\gamma_t,u-},z)\tilde N(du,dz)\Bigg|^2\Bigg]\\
&\quad+\mathbb{E}\left[\left|\int_t^{\bar t}b(u,X_{\gamma_t,u-})du+\int_t^{\bar t}\sigma(u,X_{\gamma_t,u-})dW(u)+\int_t^{\bar t}\int_{\mathbb{R}^l}\beta(u,X_{\gamma_t,u-},z)\tilde N(du,dz)\right|^2\right]\Bigg\}
\end{aligned}
$$

and equation (6.11) and Gronwall's lemma yield

$$
\mathbb{E}\left[\sup_{s\in[\bar t,T]}\left|X_{\gamma_t,T}(s)-X_{\bar\gamma_t,T}(s)\right|^2\right]\le C\left(\sup_{s\in[0,\bar t]}|\gamma_t(s)-\bar\gamma_t(s)|^2+(\bar t-t)(1+\sup_{s\in[0,t]}|\gamma_t(s)|^2)e^{C(\bar t-t)}\right). \qquad (6.12)
$$

Finally, equations (6.10) and (6.12) provide

$$
\mathbb{E}\left[\left\|X_{\gamma_t,T}-X_{\bar\gamma_t,T}\right\|_D^2\right]\le C\left(\sup_{s\in[0,\bar t]}|\gamma_t(s)-\bar\gamma_t(s)|^2+(\bar t-t)(1+\sup_{s\in[0,t]}|\gamma_t(s)|^2)e^{C(\bar t-t)}\right).
$$

□

Corollary 6.4. *Let (H0) - (H7) hold. For any $\gamma,\bar\gamma\in D([0,T],\mathbb{R}^d)$, $t,\bar t\in[0,T]$ with $t\le\bar t$, let $(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t})\in\mathbb{H}^2[t,T]$ and $(Y_{\bar\gamma_t},Z_{\bar\gamma_t},U_{\bar\gamma_t})\in\mathbb{H}^2[\bar t,T]$ be solutions to path-dependent BSDEs with jumps (6.2). Then we have the estimate*

$$
\begin{aligned}
&\left\|\left(Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t}\right)-\left(Y_{\bar\gamma_t},Z_{\bar\gamma_t},U_{\bar\gamma_t}\right)\right\|^2_{\mathbb{H}^2[\bar t,T]}\\
&\le C\left(\sup_{s\in[0,\bar t]}|\gamma_t(s)-\bar\gamma_t(s)|^2+(\bar t-t)\left(1+\sup_{s\in[0,t]}|\gamma_t(s)|^2\right)e^{C(\bar t-t)}\right). \qquad (6.13)
\end{aligned}
$$

Proof. This result follows with assumptions (H6) - (H7) and equation (3.3). □

Now, we define a function $\psi:[0,T]\times D([0,T],\mathbb{R}^d)\to\mathbb{R}$ with $\psi(t,\gamma)=Y_{\gamma_t}(t)$. In particular, it is

$$
\psi(t,\gamma)=Y_{\gamma_t}(t)=\mathbb{E}\left[\Phi(X_{\gamma_t,T})+\int_t^T f\left(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),\int_{\mathbb{R}^l}U_{\gamma_t}(r,z)\delta(r,z)\nu(dz)\right)dr\right].
$$

Corollary 6.5. *Under assumptions (H0) - (H7) the function $\psi:[0,T]\times D([0,T],\mathbb{R}^d)$ with $\psi(t,\gamma)=Y_{\gamma_t}(t)$ is continuous in (t,γ) and for a constant $C>0$ it is*

$$
|\psi(t,\gamma)|^2\le C\left(1+\sup_{s\in[0,t]}|\gamma_t(s)|^2\right)e^{C(T-t)}. \qquad (6.14)
$$

For $\gamma, \bar\gamma \in D([0,T],\mathbb{R}^d)$ and $t,\bar t \in [0,T]$ with $t \le \bar t$ we get the following estimate

$$|\psi(t,\gamma)-\psi(\bar t,\bar\gamma)|^2 \le C\left(\|\gamma_t-\bar\gamma_{\bar t}\|_D^2+(\bar t-t)\Big(1+\sup_{s\in[0,\bar t]}|\bar\gamma_{\bar t}(s)|^2+\sup_{s\in[0,t]}|\gamma_t(s)|^2\Big)\right).$$

Proof. Corollary 6.2 yields

$$|\psi(t,\gamma)|^2 = |Y_{\gamma_t}(t)|^2 \le \mathbb{E}\left[\sup_{s\in[t,T]}|Y_{\gamma_t}(s)|^2\right] \le C\left(1+\sup_{s\in[0,t]}|\gamma_t(s)|^2\right)e^{C(T-t)}. \tag{6.15}$$

Consider the parameterized path-dependent BSDE

$$\begin{aligned}Y_{\gamma_t}(s) = {}& \Phi(X_{\gamma_t,T})+\int_s^T \mathbb{1}_{[t,T]}(r)f\left(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),\int_{\mathbb{R}^l}U_{\gamma_t}(r,z)\delta(r,z)\nu(dz)\right)dr\\ &-\int_s^T Z_{\gamma_t}(r)dW(r)-\int_s^T\int_{\mathbb{R}^l}U_{\gamma_t}(r,z)\tilde N(dr,dz), \qquad s\in[0,T],\end{aligned}$$

with $Y_{\gamma_t}(s)=Y_{\gamma_t}(t)=\psi(t,\gamma)$ and $Z_{\gamma_t}(s)=0$, $U_{\gamma_t}(s,z)=0$ for $s\in[0,t)$, $z\in\mathbb{R}^l$. It is for $t,\bar t\in[0,T]$ and $\gamma,\bar\gamma\in D([0,T],\mathbb{R}^d)$

$$|\psi(t,\gamma)-\psi(\bar t,\bar\gamma)|^2 = |Y_{\gamma_t}(t)-Y_{\bar\gamma_{\bar t}}(\bar t)|^2 = |Y_{\gamma_t}(0)-Y_{\bar\gamma_{\bar t}}(0)|^2 \le \mathbb{E}\left[\sup_{s\in[0,T]}|Y_{\gamma_t}(s)-Y_{\bar\gamma_{\bar t}}(s)|^2\right].$$

With standard techniques we obtain for $t\le\bar t$

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\in[0,T]}|Y_{\gamma_t}(s)-Y_{\bar\gamma_{\bar t}}(s)|^2\right]\\ &\le C\mathbb{E}\Bigg[|\Phi(X_{\gamma_t,T})-\Phi(X_{\bar\gamma_{\bar t},T})|^2+2\int_{\bar t}^T\Bigg|f\left(r,X_{\gamma_t,r-},Y_{\bar\gamma_{\bar t}}(r-),Z_{\bar\gamma_{\bar t}}(r),\int_{\mathbb{R}^l}U_{\bar\gamma_{\bar t}}(r,z)\delta(r,z)\nu(dz)\right)\\ &\quad -f\left(r,X_{\bar\gamma_{\bar t},r-},Y_{\bar\gamma_{\bar t}}(r-),Z_{\bar\gamma_{\bar t}}(r),\int_{\mathbb{R}^l}U_{\bar\gamma_{\bar t}}(r,z)\delta(r,z)\nu(dz)\right)\Bigg|^2dr+4\int_t^{\bar t}\left|f_0(r,X_{\gamma_t,r-})\right|^2dr\\ &\quad +4\int_t^{\bar t}\left|f\left(r,X_{\gamma_t,r-},Y_{\bar\gamma_{\bar t}}(r-),Z_{\bar\gamma_{\bar t}}(r),\int_{\mathbb{R}^l}U_{\bar\gamma_{\bar t}}(r,z)\delta(r,z)\nu(dz)\right)-f_0(r,X_{\gamma_t,r-})\right|^2dr\Bigg].\end{aligned}$$

With assumptions (H3), (H5), (H6) and (H7) we get

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\in[0,T]}|Y_{\gamma_t}(s)-Y_{\bar\gamma_{\bar t}}(s)|^2\right]\\ &\le C\mathbb{E}\Bigg[\|X_{\gamma_t,T}-X_{\bar\gamma_{\bar t},T}\|_D^2+2\int_{\bar t}^T K(r)\|X_{\gamma_t,r-}-X_{\bar\gamma_{\bar t},r-}\|_D^2dr+4\int_t^{\bar t}K(1+\|X_{\gamma_t,r-}\|^2)dr\\ &\quad +4\int_t^{\bar t}K(r)|Y_{\bar\gamma_{\bar t}}(r-)|^2dr+4\int_t^{\bar t}K(r)\|Z_{\bar\gamma_{\bar t}}(r)\|^2dr+4K\int_t^{\bar t}K(r)\left|\int_{\mathbb{R}^l}U_{\bar\gamma_{\bar t}}(r,z)\nu(dz)\right|^2dr\Bigg]\end{aligned}$$

Since $Y_{\bar\gamma_{\bar t}}(r)=Y_{\bar\gamma_{\bar t}}(\bar t)=\psi(\bar t,\bar\gamma)$ and $Z_{\bar\gamma_{\bar t}}(r)=0,=U_{\bar\gamma_{\bar t}}(r,z)=0$ for $r<\bar t$ we obtain with (6.15) and (6.4)

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\in[0,T]}|Y_{\gamma_t}(s)-Y_{\bar\gamma_{\bar t}}(s)|^2\right]\\ &\le C\left\{\mathbb{E}\left[\|X_{\gamma_t,T}-X_{\bar\gamma_{\bar t},T}\|_D^2\right]\le C\left\{\mathbb{E}\left[\|X_{\gamma_t,T}-X_{\bar\gamma_{\bar t},T}\|_D^2\right]+(\bar t-t)\Big(1+\sup_{s\in[0,\bar t]}|\bar\gamma_{\bar t}(s)|^2+\sup_{s\in[0,t]}|\gamma_t(s)|^2\Big)\right\}.\right.\end{aligned}$$

With Theorem 6.3 we get

$$|\psi(t,\gamma)-\psi(\bar t,\bar\gamma)|^2 \le \mathbb{E}\left[\sup_{s\in[0,T]}|Y_{\gamma_t}(s)-Y_{\bar\gamma_{\bar t}}(s)|^2\right]$$
$$\le C\left(\|\gamma_t-\bar\gamma_{\bar t}\|_D^2+(\bar t-t)\Big(1+\sup_{s\in[0,\bar t]}|\bar\gamma_{\bar t}(s)|^2+\sup_{s\in[0,t]}|\gamma_t(s)|^2\Big)\right).$$

This estimate yields continuity of ψ for $(t,\gamma)\in[0,T]\times D([0,T],\mathbb{R}^d)$. □

For a function $\psi\in C^{1,2}([0,T]\times D([0,T],\mathbb{R}^d);\mathbb{R})$ we define the operator $\mathscr{A}$ by

$$(\mathscr{A}_s\psi)(s,x):=\frac12\sum_{i=1}^d\sum_{k=1}^d a_{ik}(s,x,x)D^2_{ik}\psi(s,x;[s,T])+\sum_{i=1}^d b_i(s,x)D^1_i\psi(s,x;[s,T])$$
$$+\int_{\mathbb{R}^l}\left[\psi(s,x^{\beta(s,x,z)})-\psi(s,x)-\sum_{i=1}^d\beta_i(s,x,z)D^1_i\psi(s,x;[s,T])\right]\nu(dz) \tag{6.16}$$

and we consider a path-dependent partial integro-differential equation (PPIDE) of the form

$$\begin{aligned}-\frac{\partial}{\partial s}\psi(s,x)&=\mathscr{A}_s\psi(s,x)\\&\quad+f\Big(s,x,\psi(s,x),D^1\psi(s,x;[s,T])\sigma(s,x),\int_{\mathbb{R}^l}[\psi(s,x^{\beta(s,x,z)})-\psi(s,x)]\delta(s,z)\nu(dz)\Big),\\ \psi(T,x)&=\Phi(x),\quad x\in D([0,T],\mathbb{R}^d).\end{aligned}\tag{6.17}$$

The following definition can be found in Barles et al. (1997) and Delong (2013), where we added boundedness and the Lipschitz condition.

Definition 6.6. *We say that $\psi\in C([0,T]\times D([0,T],\mathbb{R}^d);\mathbb{R})$ is*

(1) a viscosity subsolution to (6.17) if

$$\psi(T,x)\le\Phi(x),\quad x\in D([0,T],\mathbb{R}^d),$$

and for any $\varphi\in C^{1,2}_{b,Lip}([0,T]\times D([0,T],\mathbb{R}^d);\mathbb{R})$, whenever $(\bar t,\bar x)\in[0,T]\times D([0,T],\mathbb{R}^d)$ is a global maximum point of $\psi-\varphi$, we have

$$\begin{aligned}&-\frac{d}{dt}\varphi(\bar t,\bar x)-\mathscr{A}_t\varphi(\bar t,\bar x)\\&-f\left(\bar t,\bar x,\psi(\bar t,\bar x),D^1\varphi(\bar t,\bar x;[\bar t,T])\sigma(\bar t,\bar x),\int_{\mathbb{R}^l}[\varphi(\bar t,\bar x^{\beta(\bar t,\bar x,z)})-\varphi(\bar t,\bar x)]\delta(\bar t,z)\nu(dz)\right)\le0.\end{aligned}\tag{6.18}$$

(2) a viscosity supersolution to (6.17) if

$$\psi(T,x)\ge\Phi(x),\quad x\in D([0,T],\mathbb{R}^d),$$

and for any $\varphi\in C^{1,2}_{b,Lip}([0,T]\times D([0,T],\mathbb{R}^d);\mathbb{R})$, whenever $(\bar t,\bar x)\in[0,T]\times D([0,T],\mathbb{R}^d)$ is a global minimum point of $\psi-\varphi$, we have

$$\begin{aligned}&-\frac{d}{dt}\varphi(\bar t,\bar x)-\mathscr{A}_t\varphi(\bar t,\bar x)\\&-f\left(\bar t,\bar x,\psi(\bar t,\bar x),D^1\varphi(\bar t,\bar x;[\bar t,T])\sigma(\bar t,\bar x),\int_{\mathbb{R}^l}[\varphi(\bar t,\bar x^{\beta(\bar t,\bar x,z)})-\varphi(\bar t,\bar x)]\delta(\bar t,z)\nu(dz)\right)\ge0.\end{aligned}\tag{6.19}$$

(3) is a viscosity solution to (6.17) if it is both viscosity sub and supersolution to (6.17).

The definition of a viscosity subsolution (supersolution) is different to the definition in Keller (2016). Here, the testfunctions have to be Lipschitz continuous, bounded and have to satisfy inequality (6.18) (resp. (6.19)) only in the maximum point (resp. in the minimum point) of the difference $\psi-\varphi$, whereas in Keller (2016) the class of testfunctions depends on conditions formulated in terms of probability measures.

Theorem 6.7. *Assume that (H0) - (H7) hold. Let $(X_{\gamma_t},Y_{\gamma_t},Z_{\gamma_t},U_{\gamma_t})$ be the unique solution to FBSDE (6.1), (6.2). The function $\psi(t,\gamma)=Y_{\gamma_t}(t)$ is a viscosity solution to PPIDE (6.17).*

Proof. We use techniques of Barles et al. (1997). First we show that ψ is a viscosity subsolution to PPIDE (6.17).

Due to Corollary 6.5 ψ is in $C([0,T]\times D([0,T],\mathbb{R}^d);\mathbb{R})$. Furthermore, it is for any $\gamma\in D([0,T],\mathbb{R}^d)$

$$\psi(T,\gamma)=Y_{\gamma_T}(T)=\Phi(X_{\gamma_T,T})=\Phi(\gamma_T)=\Phi(\gamma)$$

and the function ψ satisfies the boundary condition.

Let $\varphi\in C^{1,2}_{b,Lip}([0,T]\times D([0,T],\mathbb{R}^d);\mathbb{R})$ and choose $(t,\gamma)\in[0,T]\times D([0,T],\mathbb{R}^d)$ such that $\varphi(t,\gamma)=\psi(t,\gamma)$ and $\varphi>\psi$ for $(\bar t,\bar\gamma)\in[0,T]\times D([0,T],\mathbb{R}^d)$ with $t\neq\bar t$, $\gamma\neq\bar\gamma$. this implies that (t,γ) is a global maximum point of $\psi-\varphi$.

For any $h\in\mathbb{R}^+$, where $t\le s\le t+h\le T$, it is with (6.2)

$$\begin{aligned}Y_{\gamma_t}(t+h)=Y_{\gamma_t}(s)&-\int_s^{t+h} f\left(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),\int_{\mathbb{R}^l}U_{\gamma_t}(r,z)\delta(r,z)\nu(dz)\right)dr\\&+\int_s^{t+h}Z_{\gamma_t}(r)dW(r)+\int_s^{t+h}\int_{\mathbb{R}^l}U_{\gamma_t}(r,z)\tilde N(dr,dz)\end{aligned}$$

and for any $\tilde\gamma\in D([0,T],\mathbb{R}^d)$ we obtain

$$\begin{aligned}\psi(t+h,\tilde\gamma)&=Y_{\tilde\gamma_{t+h}}(t+h)\\&=\Phi(X_{\tilde\gamma_{t+h},T})+\int_{t+h}^T f\left(r,X_{\tilde\gamma_{t+h},r-},Y_{\tilde\gamma_{t+h}}(r-),Z_{\tilde\gamma_{t+h}}(r),\int_{\mathbb{R}^l}U_{\tilde\gamma_{t+h}}(r,z)\delta(r,z)\nu(dz)\right)dr\\&\quad-\int_{t+h}^T Z_{\tilde\gamma_{t+h}}(r)dW(r)-\int_{t+h}^T\int_{\mathbb{R}^l}U_{\tilde\gamma_{t+h}}(r,z)\tilde N(dr,dz).\end{aligned}$$

With Corollary 6.4 it is

$$\begin{aligned}\mathbb{E}\left[|Y_{\gamma_t}(t+h)-\psi(t+h,\tilde\gamma)|^2\right]&=\mathbb{E}\left[|Y_{\gamma_t}(t+h)-Y_{\tilde\gamma_{t+h}}(t+h)|^2\right]\\&\le\mathbb{E}\left[\sup_{s\in[t+h,T]}|Y_{\gamma_t}(s)-Y_{\tilde\gamma_{t+h}}(s)|^2\right]\\&\le C\left(\sup_{s\in[0,t+h]}|\gamma_t(s)-\tilde\gamma_{t+h}(s)|^2+h\left(1+\sup_{s\in[0,t]}|\gamma_t(s)|^2\right)e^{Ch}\right).\end{aligned}$$

In particular, for $\tilde\gamma=X_{\gamma_t,t+h}$ this yields

$$\begin{aligned}&\mathbb{E}\left[|Y_{\gamma_t}(t+h)-\psi(t+h,X_{\gamma_t,t+h})|^2\right]\\&\le C\mathbb{E}\left[\sup_{s\in[0,t+h]}|X_{\gamma_t}(s)-X_{\gamma_t}(t)|^2+h\left(1+\sup_{s\in[0,t]}|\gamma_t(s)|^2\right)e^{Ch}\right]\\&\le Ch\left(1+\sup_{s\in[0,t]}|\gamma_t(s)|^2\right)e^{Ch}\\&\to 0,\qquad\text{for } h\to 0,\end{aligned}$$

where we use that

$$\sup_{s\in[0,t+h]}|\gamma_t(s)-X_{\gamma_t,t+h}(s)|^2=\sup_{s\in[t,t+h]}|X_{\gamma_t}(s)-X_{\gamma_t}(t)|^2.$$

Furthermore, we obtain for $s \in [t, t+h]$

$$
\begin{aligned}
Y_{\gamma_t}(s) =& \psi(t+h, X_{\gamma_t, t+h}) + \varepsilon_h + \int_s^{t+h} f\left(r, X_{\gamma_t, r-}, Y_{\gamma_t}(r-), Z_{\gamma_t}(r), \int_{\mathbb{R}^l} U_{\gamma_t}(r,z)\delta(r,z)\nu(dz)\right) dr \\
&- \int_s^{t+h} Z_{\gamma_t}(r) dW(r) - \int_s^{t+h} \int_{\mathbb{R}^l} U_{\gamma_t}(r,z)\tilde{N}(dr,dz),
\end{aligned}
$$

with $\varepsilon_h := Y_{\gamma_t}(t+h) - \psi(t+h, X_{\gamma_t, t+h})$.

Consider a path-dependent BSDE of this form

$$
\begin{aligned}
\bar{Y}_{\gamma_t}(s) =& \varphi(t+h, X_{\gamma_t, t+h}) + \int_s^{t+h} f\left(r, X_{\gamma_t, r-}, \bar{Y}_{\gamma_t}(r-), \bar{Z}_{\gamma_t}(r), \int_{\mathbb{R}^l} \bar{U}_{\gamma_t}(r,z)\delta(r,z)\nu(dz)\right) dr \\
&- \int_s^{t+h} \bar{Z}_{\gamma_t}(r) dW(r) - \int_s^{t+h} \int_{\mathbb{R}^l} \bar{U}_{\gamma_t}(r,z)\tilde{N}(dr,dz), \qquad t \le s \le t+h \le T. \qquad (6.20)
\end{aligned}
$$

Since $\varphi \in C^{1,2}_{b,Lip}([t,T] \times D([0,T], \mathbb{R}^d); \mathbb{R})$, Theorem 3.3 yields that there exist a unique solution $(\bar{Y}_{\gamma_t}, \bar{Z}_{\gamma_t}, \bar{U}_{\gamma_t}) \in \mathbb{H}^2[t, t+h]$ to BSDE (6.20). Moreover, $\varphi(t+h, X_{\gamma_t, t+h}) > \psi(t+h, X_{\gamma_t, t+h})$ and for h small enough such that

$$
\varphi(t+h, X_{\gamma_t, t+h}) \ge \psi(t+h, X_{\gamma_t, t+h}) + \varepsilon_h
$$

Corollary 4.1 provides $\bar{Y}_{\gamma_t}(s) \ge Y_{\gamma_t}(s)$ for $s \in [t, t+h]$ and in particular it is $\bar{Y}_{\gamma_t}(t) \ge \psi(t, \gamma) = \varphi(t, \gamma)$.

Now, we define

$$
\begin{aligned}
\Theta(s,x) &:= \frac{\partial \varphi}{\partial s}(s,x) + \mathscr{A}_s \varphi(s,x), \quad (s,x) \in [t,T] \times D([0,T], \mathbb{R}^d), \\
\Gamma(s,x,z) &:= \varphi(s, x^{\beta(s,x,z)}) - \varphi(s,x), \quad (s,x) \in [t,T] \times D([0,T], \mathbb{R}^d), \, z \in \mathbb{R}^l. \qquad (6.21)
\end{aligned}
$$

Since $\varphi \in C^{1,2}_{b,Lip}([t,T] \times D([0,T], \mathbb{R}^d); \mathbb{R})$ we obtain with assumption (H2)

$$
|\Theta(s,x)|^2 = \left| \frac{\partial \varphi}{\partial s}(s,x) + \mathscr{A}_s \varphi(s,x) \right|^2 \le K(1 + \sup_{u \in [0,s]} |x(u)|^2) \qquad (6.22)
$$

and

$$
|\Gamma(s,x,z)|^2 = \left| \varphi(s, x^{\beta(s,x,z)}) - \varphi(s,x) \right|^2 \le K|\beta(s,x,z)|^2 \le K(1 + \sup_{u \in [0,s]} |x(u)|^2). \qquad (6.23)
$$

With the functional Itô formula we get for $s \in [t, t+h]$

$$
\begin{aligned}
&\varphi(t+h, X_{\gamma_t, t+h}) \\
&= \varphi(s, X_{\gamma_t, s}) + \int_s^{t+h} \frac{\partial}{\partial r} \varphi(r, X_{\gamma_t, r-}) dr + \int_s^{t+h} \mathscr{A}_r \varphi(r, X_{\gamma_t, r-}) dr \\
&+ \int_s^{t+h} D^1 \varphi(r, X_{\gamma_t, r-}; [r,T]) \sigma(r, X_{\gamma_t, r-}) dW(r) + \int_s^{t+h} \int_{\mathbb{R}^l} \varphi(r, X_{\gamma_t, r-}^{\beta(r, X_{\gamma_t, r-}, z)}) - \varphi(r, X_{\gamma_t, r-}) \tilde{N}(dr,dz).
\end{aligned}
$$

In particular it is with definition (6.21)

$$
\begin{aligned}
\varphi(s, X_{\gamma_t, s}) =& \varphi(t+h, X_{\gamma_t, t+h}) - \int_s^{t+h} \Theta(r, X_{\gamma_t, r-}) dr - \int_s^{t+h} D^1 \varphi(r, X_{\gamma_t, r-}; [r,T]) \sigma(r, X_{\gamma_t, r-}) dW(r) \\
&- \int_s^{t+h} \int_{\mathbb{R}^l} \Gamma(r, X_{\gamma_t, r-}, z) \tilde{N}(dr,dz). \qquad (6.24)
\end{aligned}
$$

We define for $s \in [t, t+h]$

$$
\begin{aligned}
\hat{Y}(s) &= \bar{Y}_{\gamma_t}(s) - \varphi(s, X_{\gamma_t, s}), \\
\hat{Z}(s) &= \bar{Z}_{\gamma_t}(s) - D^1 \varphi(s, X_{\gamma_t, s}; [s,T]) \sigma(s, X_{\gamma_t, s}), \qquad (6.25) \\
\hat{U}(s,z) &= \bar{U}_{\gamma_t}(s,z) - \Gamma(s, X_{\gamma_t, s-}, z).
\end{aligned}
$$

Equations (6.20) and (6.24) yield

$$\begin{aligned}\hat{Y}(s) &= \int_s^{t+h} f\left(r, X_{\gamma_t,r-}, \bar{Y}_{\gamma_t}(r-), \bar{Z}_{\gamma_t}(r), \int_{\mathbb{R}^l} \bar{U}_{\gamma_t}(r,z)\delta(r,z)\nu(dz)\right) + \Theta(r, X_{\gamma_t,r-})dr \\ &\quad - \int_s^{t+h} \bar{Z}_{\gamma_t}(r) - D^1\varphi(r, X_{\gamma_t,r-};[r,T])\sigma(r, X_{\gamma_t,r-})dW(r) \\ &\quad - \int_s^{t+h}\int_{\mathbb{R}^l} \bar{U}_{\gamma_t}(r,z) - \Gamma(r, X_{\gamma_t,r-},z)\tilde{N}(dr,dz), \quad s\in[t,t+h],\end{aligned}$$

which implies with definition (6.25) for $s \in [t,t+h]$

$$\begin{aligned}&\hat{Y}(s) \\ &= \int_s^{t+h} \Theta(r, X_{\gamma_t,r-}) \\ &\quad + f\Bigg(r, X_{\gamma_t,r-}, \hat{Y}(r-) + \varphi(r, X_{\gamma_t,r-}), \hat{Z}(r) + D^1\varphi(r, X_{\gamma_t,r-};[r,T])\sigma(r, X_{\gamma_t,r-}), \\ &\qquad \int_{\mathbb{R}^l} \left(\hat{U}(r,z) + \Gamma(r, X_{\gamma_t,r-},z)\right)\delta(r,z)\nu(dz)\Bigg)dr - \int_s^{t+h} \hat{Z}(r)dW(r) - \int_s^{t+h}\int_{\mathbb{R}^l} \hat{U}(r,z)\tilde{N}(dr,dz). \end{aligned} \tag{6.26}$$

$(\hat{Y}, \hat{Z}, \hat{U}) \in \mathbb{H}^2[t,t+h]$ is the unique solution to BSDE (6.26).

With standard techniques we get

$$\begin{aligned}&\mathbb{E}\left[|\hat{Y}(s)|^2\right] + \mathbb{E}\left[\int_s^{t+h} \|\hat{Z}(r)\|^2 dr\right] + \mathbb{E}\left[\int_s^{t+h}\int_{\mathbb{R}^l} |\hat{U}(r,z)|^2\nu(dz)dr\right] \\ &\le 2\mathbb{E}\Bigg[\int_s^{t+h} |\hat{Y}(r-)|\cdot\Bigg|\Theta(r, X_{\gamma_t,r-}) + \\ &\quad f\Bigg(r, X_{\gamma_t,r-}, \hat{Y}(r-) + \varphi(r, X_{\gamma_t,r-}), \hat{Z}(r) + D^1\varphi(r, X_{\gamma_t,r-})\sigma(r, X_{\gamma_t,r-}), \\ &\qquad \int_{\mathbb{R}^l} \left(\hat{U}(r,z) + \Gamma(r, X_{\gamma_t,r-},z)\right)\delta(r,z)\nu(dz)\Bigg)\Bigg| dr\Bigg] \\ &\le 2\mathbb{E}\Bigg[\int_s^{t+h} |\hat{Y}(r-)|\cdot\Bigg(\left|\Theta(r, X_{\gamma_t,r-})\right| + \left|f_0\left(r, X_{\gamma_t,r-}\right)\right| \\ &\quad + \Bigg|f\left(r, X_{\gamma_t,r-}, \varphi(r, X_{\gamma_t,r-}), D^1\varphi(r, X_{\gamma_t,r-})\sigma(r, X_{\gamma_t,r-}), \int_{\mathbb{R}^l} \Gamma(r, X_{\gamma_t,r-},z)\delta(r,z)\nu(dz)\right) \\ &\quad - f_0\left(r, X_{\gamma_t,r-}\right)\Bigg|\Bigg)dr\Bigg] \\ &\quad + 2K\mathbb{E}\left[\int_s^{t+h} |\hat{Y}(r-)|\cdot\left(|\hat{Y}(r-)| + \|\hat{Z}(r)\| + \int_{\mathbb{R}^l} |\hat{U}(r,z)|\delta(r,z)\nu(dz)\right)dr\right].\end{aligned}$$

Since φ and its derivatives are bounded we obtain with help of equations (6.22), (6.23) and assumptions (H2), (H3), (H7)

$$\begin{aligned}&\mathbb{E}\left[|\hat{Y}(s)|^2\right] + \mathbb{E}\left[\int_s^{t+h} \|\hat{Z}(r)\|^2 dr\right] + \mathbb{E}\left[\int_s^{t+h}\int_{\mathbb{R}^l} |\hat{U}(r,z)|^2\nu(dz)dr\right] \\ &\le 2K\mathbb{E}\left[\int_s^{t+h} |\hat{Y}(r-)|\cdot\left(1 + \sup_{u\in[0,r]} |X_{\gamma_t,r-}(u)|\right)dr\right] \\ &\quad + 2K\mathbb{E}\left[\int_s^{t+h} |\hat{Y}(r-)|\cdot\left(|\hat{Y}(r-)| + \|\hat{Z}(r)\| + \int_{\mathbb{R}^l} |\hat{U}(r,z)|\delta(r,z)\nu(dz)\right)dr\right].\end{aligned}$$

Cauchy-Schwarz inequality and $2|\langle u,v\rangle| \le \frac{1}{\rho}|u|^2+\rho|v|^2$, $\rho>0$ yields

$$\begin{aligned}
&\mathbb{E}\left[|\hat{Y}(s)|^2\right]+\mathbb{E}\left[\int_s^{t+h}\|\hat{Z}(r)\|^2dr\right]+\mathbb{E}\left[\int_s^{t+h}\int_{\mathbb{R}^l}|\hat{U}(r,z)|^2\nu(dz)dr\right]\\
&\le 2K\mathbb{E}\left[\int_s^{t+h}|\hat{Y}(r-)|\cdot\left(1+|\gamma_t(t)|+\sup_{u\in[0,r]}|X_{\gamma_t,r-}(u)-\gamma_t(t)|\right)dr\right]\\
&\quad+3K\mathbb{E}\left[\int_s^{t+h}\frac{\rho^2+1}{\rho}|\hat{Y}(r-)|^2+\frac{1}{\rho}\left(\|\hat{Z}(r)\|^2+\int_{\mathbb{R}^l}|\hat{U}(r,z)|^2\nu(dz)\int_{\mathbb{R}^l}|\delta(r,z)|^2\nu(dz)\right)dr\right]\\
&\le C\mathbb{E}\left[\int_s^{t+h}|\hat{Y}(r-)|+|\hat{Y}(r-)|^2+\sup_{u\in[0,r]}|X_{\gamma_t,r-}(u)-\gamma_t(t)|^2dr\right]\\
&\quad+\frac{C}{\rho}\mathbb{E}\left[\int_s^{t+h}\|\hat{Z}(r)\|^2dr+\int_s^{t+h}\int_{\mathbb{R}^l}|\hat{U}(r,z)|^2\nu(dz)dr\right].
\end{aligned}$$

Thus, it follows for ρ big enough

$$\begin{aligned}
&\mathbb{E}\left[|\hat{Y}(s)|^2\right]+\mathbb{E}\left[\int_s^{t+h}\|\hat{Z}(r)\|^2dr\right]+\mathbb{E}\left[\int_s^{t+h}\int_{\mathbb{R}^l}|\hat{U}(r,z)|^2\nu(dz)dr\right]\\
&\le C\mathbb{E}\left[\int_s^{t+h}|\hat{Y}(r-)|+|\hat{Y}(r-)|^2dr\right]+C\int_s^{t+h}h\left(1+\sup_{u\in[0,t]}|\gamma_t(u)|^2\right)e^{Ch}dr.
\end{aligned}\tag{6.27}$$

In particular, it is for $s\in[t,t+h]$ with $|y|\le 1+|y|^2$

$$\mathbb{E}\left[|\hat{Y}(s)|^2\right]\le C(h+h^2)\left(1+\sup_{u\in[0,t]}|\gamma_t(u)|^2\right)e^{Ch}+C\mathbb{E}\left[\int_s^{t+h}|\hat{Y}(r-)|^2dr\right].$$

Gronwall's lemma provides for sufficiently small $h>0$

$$\mathbb{E}\left[|\hat{Y}(s)|^2\right]\le C(h+h^2)\left(1+\sup_{u\in[0,t]}|\gamma_t(u)|^2\right)e^{2Ch}\le Ch\left(1+\sup_{u\in[0,t]}|\gamma_t(u)|^2\right).\tag{6.28}$$

Jensen's inequality implies

$$\mathbb{E}\left[|\hat{Y}(s)|\right]\le Ch^{\frac{1}{2}}\left(1+\sup_{u\in[0,t]}|\gamma_t(u)|\right)$$

and together with equation (6.27) we get

$$\begin{aligned}
&\mathbb{E}\left[\int_t^{t+h}\|\hat{Z}(r)\|^2dr\right]+\mathbb{E}\left[\int_t^{t+h}\int_{\mathbb{R}^l}|\hat{U}(r,z)|^2\nu(dz)dr\right]\\
&\le C\left(h^{\frac{3}{2}}\left(1+\sup_{u\in[0,t]}|\gamma_t(u)|\right)+h^2\left(1+\sup_{u\in[0,t]}|\gamma_t(u)|^2\right)\left(1+e^{Ch}\right)\right).
\end{aligned}\tag{6.29}$$

Now, we suppose that

$$\begin{aligned}
&-\frac{\partial}{\partial t}\varphi(t,\gamma)-\mathscr{A}_t\varphi(t,\gamma)\\
&-f\left(t,\gamma,\psi(t,\gamma),D^1\varphi(t,\gamma;[t,T])\sigma(t,\gamma),\int_{\mathbb{R}^l}[\varphi(t,\gamma^{\beta(t,\gamma,z)})-\varphi(t,\gamma)]\delta(t,z)\nu(dz)\right)>0.
\end{aligned}\tag{6.30}$$

Therefore, there exist some $\alpha>0$ and some $h_0>0$ such that for all $t-h_0\le\bar{t}\le t+h_0$ and for all $\bar{\gamma}\in D([0,T],\mathbb{R}^d)$ with $\|\gamma-\bar{\gamma}\|_D^2\le h_0$

$$\begin{aligned}
&-\frac{\partial}{\partial t}\varphi(\bar{t},\bar{\gamma})-\mathscr{A}_t\varphi(\bar{t},\bar{\gamma})\\
&-f\left(\bar{t},\bar{\gamma},\psi(\bar{t},\bar{\gamma}),D^1\varphi(\bar{t},\bar{\gamma};[\bar{t},T])\sigma(\bar{t},\bar{\gamma}),\int_{\mathbb{R}^l}[\varphi(\bar{t},\bar{\gamma}^{\beta(\bar{t},\bar{\gamma},z)})-\varphi(\bar{t},\bar{\gamma})]\delta(\bar{t},z)\nu(dz)\right)\ge\alpha.
\end{aligned}$$

This implies for $h < h_0$

$$\xi_h := \frac{1}{h}\mathbb{E}\Bigg[\int_t^{t+h} \Theta(r, X_{\gamma_t,r-}) + f\Bigg(r, X_{\gamma_t,r-}, \varphi(r, X_{\gamma_t,r-}), D^1\varphi(r, X_{\gamma_t,r-};[r,T])\sigma(r, X_{\gamma_t,r-}),$$
$$\int_{\mathbb{R}^l} [\varphi(r, X_{\gamma_t,r-}^{\beta(r,X_{\gamma_t,r-},z)}) - \varphi(r, X_{\gamma_t,r-})]\delta(r,z)\nu(dz)\Bigg)dr\Bigg] \leq -\alpha.$$

Since $\hat{Y}(s) = \bar{Y}_{\gamma_t}(s) - \varphi(s, X_{\gamma_t,s})$ for $s \in [t, t+h]$ and $\bar{Y}_{\gamma_t}(t) \geq \varphi(t, \gamma)$ it is $\hat{Y}(t) \geq 0$ and we get

$$\begin{aligned}
0 &\leq \frac{1}{h}\hat{Y}(t) \\
&= \frac{1}{h}\mathbb{E}\Bigg[\int_t^{t+h} \Theta(r, X_{\gamma_t,r-}) + f\Bigg(r, X_{\gamma_t,r-}, \hat{Y}(r-) + \varphi(r, X_{\gamma_t,r-}), \hat{Z}(r) + D^1\varphi(r, X_{\gamma_t,r-};[r,T])\sigma(r, X_{\gamma_t,r-}), \\
&\qquad \int_{\mathbb{R}^l} \left(\hat{U}(r,z) + \Gamma(r, X_{\gamma_t,r-}, z)\right)\delta(r,z)\nu(dz)\Bigg)dr\Bigg].
\end{aligned}$$

This implies

$$\begin{aligned}
\alpha &\leq \left|\frac{1}{h}\hat{Y}(t) - \xi_h\right| \\
&\leq K \sup_{s\in[t,t+h]} \mathbb{E}\left[|\hat{Y}(s)|\right] + K\mathbb{E}\left[\frac{1}{h}\int_t^{t+h} \|\hat{Z}(r)\|^2 dr\right]^{\frac{1}{2}} + K\mathbb{E}\left[\frac{1}{h}\int_t^{t+h}\int_{\mathbb{R}^l} |\hat{U}(r,z)|^2\nu(dz)dr\right]^{\frac{1}{2}}.
\end{aligned}$$

With equations (6.28) and (6.29) we get

$$\alpha \leq Ch^{\frac{1}{2}}\left(1 + \sup_{s\in[0,t]} |\gamma_t(s)|^2\right) + 2C\left(h^{\frac{1}{4}}\left(1 + \sup_{u\in[0,t]} |\gamma_t(u)|\right) + h^{\frac{1}{2}}\left(1 + \sup_{u\in[0,t]} |\gamma_t(u)|\right)\left(1 + e^{Ch}\right)^{\frac{1}{2}}\right).$$

For sufficiently small $h > 0$ equation (6.30) cannot be true and φ is a subsolution to (6.17). Analogous you can show that φ is also a supersolution to (6.17).

□

7 Functional Feynman-Kac Theorem for path-dependent FBSDEs

According to Theorem 3.1 and Corollary 3.3 in Kromer, Overbeck, and Röder (2015) we know that (6.1) has a unique strong solution.

Now, we are able to prove the functional Feynman-Kac theorem that is based on the path-dependent FBSDE (6.1)-(6.2).

Theorem 7.1. *Suppose that properties (H0)-(H2) hold, $\psi \in C^{1,2}([0,T] \times D([0,T], \mathbb{R}^d); \mathbb{R}^m)$ satisfies the assumptions of Theorem 2.3 and the PPIDE (6.17) as well as the linear growth conditions for ψ and $D^1\psi$*

$$\sup_{s\in[0,T]} |\psi(s,x)|^2 \leq M(1 + \sup_{s\in[0,T]} |x(s)|^2); \quad x \in D([0,T], \mathbb{R}^d), \tag{7.1}$$

$$\sup_{s\in[0,T]} \|D^1\psi(s,x;[s,T])\|^2 \leq M(1 + \sup_{s\in[0,T]} |x(s)|^2); \quad x \in D([0,T], \mathbb{R}^d). \tag{7.2}$$

Then

$$\begin{aligned}
Y_{\gamma_t}(s) &= \psi(s, X_{\gamma_t,s}) \in \mathbb{R}^m, \\
Z_{\gamma_t}(s) &= D^1\psi(s, X_{\gamma_t,s-};[s,T])\sigma(s, X_{\gamma_t,s-}) \in \mathbb{R}^{m\times d}, \\
U_{\gamma_t}(s,z) &= \psi\left(s, X_{\gamma_t,s-}^{\beta(s,X_{\gamma_t,s-},z)}\right) - \psi(s, X_{\gamma_t,s-}) \in \mathbb{R}^m, \quad t \leq s \leq T,
\end{aligned} \tag{7.3}$$

is the unique solution to BSDE (6.2)*. Moreover, we have the representation*

$$
\begin{aligned}
Y_{\gamma_t}(s) =&\psi(s,X_{\gamma_t,s})\\
=&\mathbb{E}\left[h(X_{\gamma_t,T})+\int_s^T f\Big(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),\int_{\mathbb{R}^l}U_{\gamma_t}(r,z)\delta(r,z)\nu(dz)\Big)dr\middle|\mathscr{F}_s^t\right].
\end{aligned}
\tag{7.4}
$$

Proof. To simplify the notation, we consider the one-dimensional case where $m=d=l=1$. We apply Itô's formula from Theorem 2.3 to the function ψ and use the representation of X_{γ_t} from (6.1) to obtain

$$
\begin{aligned}
&\psi(T,X_{\gamma_t,T})\\
&=\psi(s,X_{\gamma_t,s})+\int_s^T\frac{\partial}{\partial r}\psi(r,X_{\gamma_t,r-})dr+\int_s^T D^1\psi(r,X_{\gamma_t,r-};[r,T])b(r,X_{\gamma_t,r-})dr\\
&\quad+\int_s^T D^1\psi(r,X_{\gamma_t,r-};[r,T])\sigma(r,X_{\gamma_t,r-})dW(r)\\
&\quad+\int_s^T\int_{\mathbb{R}^l}D^1\psi(r,X_{\gamma_t,r-};[r,T])\beta(r,X_{\gamma_t,r-},z)\tilde{N}(dr,dz)\\
&\quad+\frac{1}{2}\int_s^T D^2\psi(r,X_{\gamma_t,r-};[r,T])\sigma^2(r,X_{\gamma_t,r-})dr\\
&\quad+\int_s^T\int_{\mathbb{R}^l}\Big[\psi(r,X_{\gamma_t,r-}^{\beta(r,X_{\gamma_t,r-},z)})-\psi(r,X_{\gamma_t,r-})-D^1\psi(r,X_{\gamma_t,r-};[r,T])\beta(r,X_{\gamma_t,r-},z)\Big]N(dr,dz)\\
&=\psi(s,X_{\gamma_t,s})+\int_s^T\frac{\partial}{\partial r}\psi(r,X_{\gamma_t,r-})dr+\int_s^T D^1\psi(r,X_{\gamma_t,r-};[r,T])b(r,X_{\gamma_t,r-})dr\\
&\quad+\int_s^T D^1\psi(r,X_{\gamma_t,r-};[r,T])\sigma(r,X_{\gamma_t,r-})dW(r)\\
&\quad+\int_s^T\int_{\mathbb{R}^l}D^1\psi(r,X_{\gamma_t,r-};[r,T])\beta(r,X_{\gamma_t,r-},z)\tilde{N}(dr,dz)\\
&\quad+\frac{1}{2}\int_s^T D^2\psi(r,X_{\gamma_t,r-};[r,T])\sigma^2(r,X_{\gamma_t,r-})dr\\
&\quad+\int_s^T\int_{\mathbb{R}^l}\psi(r,X_{\gamma_t,r-}^{\beta(r,X_{\gamma_t,r-},z)})-\psi(r,X_{\gamma_t,r-})-D^1\psi(r,X_{\gamma_t,r-};[r,T])\beta(r,X_{\gamma_t,r-},z)\tilde{N}(dr,dz)\\
&\quad+\int_s^T\int_{\mathbb{R}^l}\psi(r,X_{\gamma_t,r-}^{\beta(r,X_{\gamma_t,r-},z)})-\psi(r,X_{\gamma_t,r-})-D^1\psi(r,X_{\gamma_t,r-};[r,T])\beta(r,X_{\gamma_t,r-},z)\nu(dz)dr.
\end{aligned}
$$

Since ψ satisfies PPIDE (6.17), the above equality leads to

$$
\begin{aligned}
h(X_{\gamma_t,T})=&\psi(s,X_{\gamma_t,s})+\int_s^T D^1\psi(r,X_{\gamma_t,r-};[r,T])\sigma(r,X_{\gamma_t,r-})dW(r)\\
&+\int_s^T\int_{\mathbb{R}^l}\Big[\psi(r,X_{\gamma_t,r-}^{\beta(r,X_{\gamma_t,r-},z)})-\psi(r,X_{\gamma_t,r-})\Big]\tilde{N}(dr,dz)\\
&-\int_s^T f\Big(r,X_{\gamma_t,r-},\psi(r,X_{\gamma_t,r-}),D^1\psi(r,X_{\gamma_t,r-};[r,T])\sigma(r,X_{\gamma_t,r-}),\\
&\qquad\int_{\mathbb{R}^l}[\psi(r,X_{\gamma_t,r-}^{\beta(r,X_{\gamma_t,r-},z)})-\psi(r,X_{\gamma_t,r-})]\delta(r,z)\nu(dz)\Big)dr.
\end{aligned}
\tag{7.5}
$$

(7.5) implies that (7.3) satisfies BSDE (6.2).

To show that Y_{γ_t} from (7.3) is in $S^2[t,T]$ we will use the linear growth condition of the function ψ in (7.1) and inequality (6.4) which holds under the conditions (H1)-(H2) on the parameters b,σ,β. From these properties it follows $\mathbb{E}[\sup_{s\in[t,T]}|Y_{\gamma_t}(s)|^2]\leq L(1+\sup_{s\in[0,T]}|\gamma_t(s)|^2)$, where the constant L depends on T,t,M and C. From (7.2) and (C2) we immediately get $Z_{\gamma_t}\in M^2[t,T]$. Finally, again from (H2) for β and (6.4) we get $U_{\gamma_t}\in L_N^2(t,T)$.

From $Z_{\gamma_t}\in M^2[t,T]$ and $U_{\gamma_t}\in L_N^2(t,T)$ it follows from Theorem 2.3.3 in Delong (2013) that the second and third summand on right hand side of equality (7.5) are càdlàg, square integrable martingales. Thus, if we apply the conditional expectation $\mathbb{E}^{t,\gamma_t}[\cdot]$ to (7.5) we get

$$
\begin{aligned}
Y_{\gamma_t}(s) &= \psi(s,X_{\gamma_t,s}) \\
&= \mathbb{E}\left[h(X_{\gamma_t,T}) + \int_s^T f\Big(r,X_{\gamma_t,r-},Y_{\gamma_t}(r-),Z_{\gamma_t}(r),\int_{\mathbb{R}^l} U_{\gamma_t}(r,z)\delta(r,z)\nu(dz)\Big)dr\middle|\mathscr{F}_s^t\right].
\end{aligned}
$$

which is the representation (7.4). □

Remark 7.2. *After finishing the first draft of this paper we learned about the paper of Christian Keller, see Keller (2016), that studies path-dependent partial integro-differential equations of type* (6.17)*, their viscosity solutions and the relation of viscosity solutions to classical solutions. Thus, we refer the reader who is interested in the conditions on ψ that guarantee existence and uniqueness of solutions to PPIDE (6.17) to this paper, and in particular to Theorem 3.19 for the existence and uniqueness result and to Theorem 3.6 for the relation of viscosity solutions to classical solutions.*

In Kromer, Overbeck, and Röder (2015) we established an approach to CVA-pricing in terms of a path-dependent PIDE that allows for CVA-pricing of path-dependent derivatives. Now, with Theorem 7.1 from Section 7 we are able to establish a connection between the path-dependent CVA-PIDE from Kromer et al. (2015) to solutions of specific BSDEs that are simplified versions of BSDE (6.2).

Remark 7.3. *We consider a path-dependent version of the market model introduced in Burgard and Kjaer (2011)*

$$
\begin{aligned}
dS_{\gamma_t^1}(t+u) &= \mu(t+u,S_{\gamma_t^1,(t+u)-})du + \sigma(t+u,S_{\gamma_t^1,(t+u)-})dW(u) \\
&\quad + \int_{\mathbb{R}} b(t+u,S_{\gamma_t^1,(t+u)-},z)\tilde{N}(du,dz), && (7.6)\\
dP^B_{\gamma_t^2}(t+u) &= r^B(t+u,P^B_{\gamma_t^2,(t+u)-})du - p^B(t+u,P^B_{\gamma_t^2,(t+u)-})dJ^B(u), && (7.7)\\
dP^C_{\gamma_t^3}(t+u) &= r^C(t+u,P^C_{\gamma_t^3,(t+u)-})du - p^C(t+u,P^C_{\gamma_t^3,(t+u)-})dJ^C(u), && (7.8)
\end{aligned}
$$

with

$$
S_{\gamma_t^1,t}(u) = \gamma_t^1(u), \quad P^C_{\gamma_t^2,t}(u) = \gamma_t^2(u), \quad P^C_{\gamma_t^3,t}(u) = \gamma_t^3(u), \tag{7.9}
$$

for $\gamma^i \in D([0,T],\mathbb{R})$, $i = 1,2,3$. We denote by γ_t the path $\gamma_t := (\gamma_t^1,\gamma_t^2,\gamma_t^3)^T \in D([0,T],\mathbb{R}^3)$ and simplify $X_{\gamma_t} := (S_{\gamma_t^1},P^B_{\gamma_t^2},P^C_{\gamma_t^3})$ with the path $X_{\gamma_t,t+u}$ defined by $X_{\gamma_t,t+u}(r) = (S_{\gamma_t^1,t+u}(r),P^B_{\gamma_t^2,t+u}(r),P^C_{\gamma_t^3,t+u}(r))$, $t,r,\in[0,T]$, $u\in[0,T-t]$. Furthermore, let the coefficients in the market model be positive and satisfy (C1)-(C2). J^B and J^C are two point processes that jump from 0 *to* 1 *on default of party B, respectively C, and we assume that the three jump processes are independent. As in Kromer et al. (2015) we can derive the following path-dependent CVA-PIDE for the CVA V*

$$
\frac{\partial}{\partial s}V\left(s,X_{\gamma_t,s-};[s,T]\right) = \mathscr{A}_s V\left(s,X_{\gamma_t,s-}\right) + g\left(s,X_{\gamma_t,s-}\right), \quad V\left(T,X_{\gamma_t,T}\right) = 0, \tag{7.10}
$$

where $\mathscr{A}$ is the generator that corresponds to our generator from (6.16)*. Now, Theorem 7.1 shows that the solution V to the path-dependent CVA-PIDE* (7.10) *is connected to BSDE*

$$
dY_{\gamma_t}(s) = -g\left(s,X_{\gamma_t,s-}\right)ds + Z_{\gamma_t}(s)dW(s) + \int_{\mathbb{R}^l} U_{\gamma_t}(s,z)\tilde{N}(ds,dz), \quad Y_{\gamma_t}(T) = 0. \tag{7.11}
$$

The connection is established by the solutions to BSDE (7.11) *which are given by*

$$
\begin{aligned}
Y_{\gamma_t}(s) &= V(s,X_{\gamma_t,s-}), && (7.12)\\
Z_{\gamma_t}(s) &= D^1V(s,X_{\gamma_t,s-};[s,T])\cdot(\sigma(s,X_{\gamma_t,s-}),0,0), \\
U_{\gamma_t}(s,z) &= V\left(s,X_{\gamma_t,s-}^{\beta(s,X_{\gamma_t,s-},z)}\right) - V(s,X_{\gamma_t,s-}), \quad t\le s\le T,
\end{aligned}
$$

where $\beta(s,X_{\gamma_t,s-},z)) = (b(s,S_{\gamma_t^1,s-},z),p^B(s,P^B_{\gamma_t^2,s-}),p^C(s,P^C_{\gamma_t^3,s-}))$.

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