A note on optimal risk sharing on L^p spaces

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Abstract

We study Pareto optimality and optimal risk sharing in terms of convex risk measures on L^p-spaces and provide a characterization result for Pareto optimality of solutions. In comparison to similar approaches that study this problem on L^∞ this setting introduces more flexibility in terms of the underlying model space. Furthermore, in our setting agents can incorporate different risk measures where some of them reflect their own preferences and others reflect requirements from regulators.
Keywords Risk sharing; Law-invariant convex risk measures; Maximum risk measure; Pareto optimality; Portfolio theory.

1 Introduction

The problem of optimal risk sharing between two or more agents has been studied in several papers. The problem formulation is very general and allows one to study interactions of agents with different preferences towards risk in various contexts. It started with the early works of [1] and [2] with applications to insurance problems and resulted in several recently published papers like [3], [4], [5] and [6]. It has applications in different areas of finance, particularly the areas of actuarial science and portfolio theory are of great importance. The problem formulation is the following: Consider m agents, each with an initial random endowment. These initial endowments add up to an aggregate risky position. Now the optimal risk sharing problem is to find an optimal allocation of this aggregate position such that the allocated risk is acceptable to each agent. Initial endowments of the agents can represent initial endowments in a stock market but equally interpretations like randomly varying water endowments or nation's quota in producing diverse pollutants are possible, see [3]. Optimality in this context in general stands for Pareto optimality. This means that there is no other allocation such that all agents are better off according to their attitude towards risk and at least one agent of them is strictly better off. The attitude towards risk of each agent is usually represented by von Neumann-Morgenstern expected utility. Existence results and characterizations of optimal risk exchanges in this framework were obtained in various papers, see [7] for an overview. In more recent works, which study the problem from a financial risk perspective, coherent or convex risk measures have taken the role of the individual risk preference functional of each agent, see for instance [6] and [8]. As demonstrated in [9] the characterization of Pareto optimal allocations in this framework reduces to the calculation of the inf-convolution of convex risk measures. The inf-convolution of convex risk measures was further studied in [8] and became the basis of studying the optimal risk sharing problem in [6], where a subdifferential characterization of Pareto optimality was derived for the optimal risk sharing problem on L^∞.

In our approach we will study the coalitional risk measure (introduced in [5] and [10] in terms of general deviation measures) which will be based on convex risk measures. This auxiliary functional will be the key to our characterization result for the proposed risk sharing problem and for Pareto optimality of the solutions. At first we will provide an existence result for solutions to our optimal risk sharing problem on L^p, $1 \leq p \leq \infty$. Then we will study the structural properties of the coalitional risk measure. Based on the corresponding properties of the underlying risk measures we will show that it is monotone, convex and cash-subadditive and thus obtains a dual representation according to the results in [11]. Finally we will derive a characterization result for Pareto optimal solutions to this problem on

L^p, $1 \leq p \leq \infty$. Thus, the main contribution of this work is, on one hand to provide an extension of the characterization result of Pareto optimality in conjunction with convex risk measures from L^∞ to L^p spaces and, on the other hand to introduce different techniques for the proofs of this result (instead of the inf-convolution of risk measures we will use the coalitional risk measure).

2 Notation and important properties of convex risk measures

A risky position will be modeled as a real valued random variable X from L^p, $1 \leq p \leq \infty$, on a non-atomic probability space $(\Omega, \mathscr{F}, \mathbb{P})$. As usual, we identify two random variables if they agree $\mathbb{P}$-a.s. Inequalities between random variables are understood in the $\mathbb{P}$-a.s. sense. We will introduce for each agent i her individual risk measure $\rho_i : L^p \to \mathbb{R} \cup \{\infty\}$, $i = 1, \dots, m$. This risk measure will represent the agents individual attitude towards risk. We will work with convex and coherent risk measures. Convex risk measures were introduced and studied in [12] and [13] based on the following properties.

(R1) $\rho(0) = 0$ and $\rho(X + r) = \rho(X) - r$ for any $X \in L^p$ and any $r \in \mathbb{R}$,

(R2) $\rho(X) \leq \rho(Y)$ for $X \geq Y$, $X, Y \in L^p$,

(R3) $\rho(\lambda X + (1-\lambda)Y) \leq \lambda \rho(X) + (1-\lambda)\rho(Y)$, for $0 \leq \lambda \leq 1$ and any $X, Y \in L^p$.

A convex risk measure with the property

(R4) $\rho(\lambda X) = \lambda \rho(X)$ for any $X \in L^p$ and $\lambda \in \mathbb{R}_+$,

is called coherent risk measure. Coherent risk measures were introduced and studied in [14]. Further properties like comonotonicity of random variables and law invariance, the Fatou and the Lebesgue property of risk measures will be needed for our main results. Law invariance of a functional $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ is formulated as follows. ρ is law-invariant, if $\rho(X) = \rho(Y)$ whenever $X, Y \in L^p$ have the same probability law. A collection $(Y_1, \dots, Y_m) \in (L^p)^m$ of m real-valued random variables on $(\Omega, \mathscr{F}, \mathbb{P})$ is comonotone if for every $(i, j) \in \{1, \dots, m\}^2$

$$(Y_i(\omega') - Y_i(\omega))(Y_j(\omega') - Y_j(\omega)) \geq 0,$$

for $\mathbb{P} \otimes \mathbb{P}$-a.e. $(\omega', \omega) \in \Omega^2$.

A functional $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ satisfies the Lebesgue property if for every uniformly bounded sequence $(X_n)_{n=1}^\infty$ tending a.s. to $X \in L^p$ we have $\rho(X) = \lim_{n\to\infty} \rho(X_n)$. ρ satisfies the Fatou property if we replace the equality in the Lebesgue property with $\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$. Additionally the concept of convex orders will be needed in the following. Let $X, Y \in L^p$, then X dominates Y in the convex order if and only if $\mathbb{E}(\varphi(X)) \geq \mathbb{E}(\varphi(Y))$ for any convex function $\varphi : \mathbb{R} \to \mathbb{R}$.

Furthermore the subdifferential of a convex function will play an important role. Here we will follow the notation from [15]. A linear functional $l : L^p \to \mathbb{R}$ is called algebraic subgradient of a convex functional ρ on L^p at $Z \in \operatorname{dom} \rho$ if

$$\rho(X') \geq \rho(Z) + l(X' - Z) \; \forall X' \in L^p. \tag{1}$$

We will denote the dual space of $L^p(\Omega, \mathscr{F}, \mathbb{P})$, $1 \leq p \leq \infty$ by $L^p(\Omega, \mathscr{F}, \mathbb{P})^*$. The dual space of $L^\infty(\Omega, \mathscr{F}, \mathbb{P})$ can be identified with $ba(\Omega, \mathscr{F}, \mathbb{P})$, the space of bounded finitely additive signed measures μ on $(\Omega, \mathscr{F})$ such that $\mathbb{P}(A) = 0$ implies $\mu(A) = 0$. The dual space of L^1 can be identified with L^∞ and the dual space of L^p for $1 < p < \infty$ is L^q, where $1/p + 1/q = 1$. If l is from the dual space of L^p, $1 \leq p \leq \infty$ and satisfies (1) we will call it subgradient of ρ. The set of all subgradients of ρ at Z will be denoted by $\partial\rho(Z)$ and we will call $\partial\rho(Z)$ the subdifferential of ρ at Z. It is said that ρ is subdifferentiable at Z if $\partial\rho(Z)$ is nonempty. Moreover, we will need the Fenchel conjugate of a proper convex risk measure on L^p which is a function α from the dual space $L^p(\Omega, \mathscr{F}, \mathbb{P})^*$ of $L^p(\Omega, \mathscr{F}, \mathbb{P})$ to $\mathbb{R} \cup \{\infty\}$ that is defined by

$$\alpha(l) = \sup_{X \in L^p(\Omega, \mathscr{F}, \mathbb{P})} \{l(X) - \rho(X)\}$$

Note that property (R1) implies that convex risk measures are proper. For convex risk measures on L^∞ this implies finiteness and continuity on L^∞. Furthermore we know that any finite convex risk measure on L^p, $1 \leq p \leq \infty$ is continuous and subdifferentiable on L^p, see Proposition 3.1 in [15] or Corollary 2.3 in [16].

Remark 2.1. *In the main part of the paper we will work with law-invariant convex risk measures. In this regard, note that Jouini et al. [17] have shown for convex functionals on L^∞ which are law-invariant and lower semicontinuous with respect to the topology induced by $\|.\|_\infty$ that these are lower semicontinuous with respect to the $\sigma(L^\infty, L^1)$-topology (see Theorem 2.2 in [17]). This implies for finite law-invariant convex risk measures on L^∞ a dual representation*

$$\rho(X) = \sup_{Z \in L^1} \{\mathbb{E}[ZX] - \alpha(Z)\} \tag{2}$$

where the supremum is formed over L^1 instead of ba. If a finite law-invariant convex risk measure (which automatically satisfies the Fatou property) additionally satisfies the stronger Lebesgue property the supremum in (2) *is attained for every $X \in L^\infty$ (see Theorem 5.2 in [17]) and it follows from* (2) *and the characterization of the subgradient of a convex risk measure ρ at X, namely*

$$Z \in \partial\rho(X) \Leftrightarrow \rho(X) = \mathbb{E}[ZX] - \alpha(Z),$$

that the subgradients of ρ at X are in L^1, i.e. $\partial\rho(X) \subset L^1$. We will use this fact in Corollary 4.3 and Corollary 4.9. With respect to the Lebesgue property of finite convex risk measures on L^p, $1 \le p < \infty$, note that these risk measures are automatically Fatou and Lebesgue continuous, see Theorem 3.1 in [16].

3 The risk sharing problem

We are interested in the following problem. Suppose there are m agents who view risk differently and who are bound to different multiple regulatory requirements. Consequently each agent i is equipped with multiple individual risk measures

$$\rho_i^1, \ldots, \rho_i^{r_i}$$

where some of them may reflect her own preferences and other are regulatory requirements. Following the motivation in [10] the agents may find different aspects of an asset to be attractive and thus they may decide to form a joint portfolio in which the share of investor i is preferred over the optimal portfolio that investor i can form alone. Then the question arises, how to divide the future payoff of cooperative portfolio X among the agents. Thus we consider divisions $Y = (Y_1, \ldots, Y_m)$, $Y_i \in L^p$, $i = 1, \ldots, m$ of X such that $\sum_{i=1}^m Y_i = X$. We will denote the set of these divisions by

$$\mathscr{A}(X) = \left\{ Y = (Y_1, \ldots, Y_m) \in (L^p)^m \,\middle|\, \sum_{i=1}^m Y_i = X \right\}$$

and by $\mathscr{C}(X) \subset \mathscr{A}(X)$ we will denote the set of random vectors from $\mathscr{A}(X)$ with comonotone components.

We will assume that each investor takes into account her individual risk measures ρ_i^j by applying cautious risk measurement methods in order to meet worst case situations and chooses to use

$$\rho_i^{max}(X) = \max\{\rho_i^1(X), \ldots, \rho_i^{r_i}(X)\}$$

as her risk measure.

Note that since we consider the maximum risk measure for each agent we allow each agent to have her individual number of different risk measures. In this way agents can incorporate any number of different risk measures where some of them reflect their own preferences and others reflect requirements from regulators. The maximum risk measure itself is a convex risk measure, independent of the number of the underlying convex risk measures. Thus, in this section on risk sharing, we will work directly with ρ_i^{max} instead of taking into account each ρ_i^j individually.

The risk sharing problem described above is usually studied in the literature in terms of the inf-convolution operator

$$\inf_{Z \in \mathscr{A}(X)} \sum_{i=1}^m \rho_i(Z_i),$$

see for instance [6] and [8] among many others. We will show that there exists another useful approach to the risk sharing problem that leads to interesting results and utilizes another operator, namely

$$\mathscr{G}_S(X) = \inf_{Z \in \mathscr{A}(X)} \max\{\rho_1^{max}(Z_1), \ldots, \rho_m^{max}(Z_m)\}. \tag{3}$$

Furthermore, we feel that the maximum operator is the more natural operator in our setting where each investor i chooses to use ρ_i^{max} as her personal cautious risk measurement method.

We will call this functional coalitional risk measure and follow in this regard a similar approach as in [5] who have studied this functional in conjunction with general deviation measures instead of convex risk measures.

We will see in the remaining part of the paper that solutions to the optimization problem (3) are strongly connected to Pareto optimality and we will provide a characterization of Pareto optimality in terms of the subdifferential of $\mathscr{G}_S$ (see Section 4).

In this section we will show that the infimum in (3) is attained for some $Z \in \mathscr{C}(X)$. Furthermore we will study the properties of coalitional risk measures. We will see that based on the corresponding properties of the underlying risk measures the coalitional risk measure is monotone, convex and cash-subadditive and thus obtains a dual representation according to the results in [11]. For the existence of solutions to (3) we will need the following results from [18] and [19].

Lemma 3.1 ([18])**.** *Let $f^n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of increasing 1-Lipschitz-continuous functions such that $f^n(0) \in [-K, K]$ for all $n \in \mathbb{N}$ where $K \ge 0$ is a constant. Then there is a* *subsequence $(f^{n_k})_{k\in\mathbb{N}}$ of $(f^n)_{n\in\mathbb{N}}$ and an increasing 1-Lipschitz-continuous function $f:\mathbb{R}\to\mathbb{R}$ such that $\lim_{k\to\infty} f^{n_k}(x)=f(x)$ for all $x\in\mathbb{R}$.*

Lemma 3.2 ([19])**.** *Let $(\Omega,\mathscr{F},\mathbb{P})$ be non-atomic, then the set of comonotone allocations of X is convex and compact in L^∞ up to zero-sum translations (which means that it can be written as $\mathscr{C}(X)=\{(\lambda_1,\dots,\lambda_m)\,|\sum_{i=1}^m\lambda_i=0\}+A_0$ with A_0 compact in $(L^\infty)^m$. In particular, the set of comonotone allocations of X is closed in L^∞.*

Note that from (A.16) in [18] we have $\rho(X)\geq\mathbb{E}[-X]+\rho(0)$ for law-invariant convex risk measures with the Fatou property. Thus for $\mathscr{G}_S$ based on such risk measures we have $\mathscr{G}_S(0)=0$. Define $\mathbb{A}=\{(f_1,\dots,f_m)|f_i:\mathbb{R}\to\mathbb{R}$ are increasing, $\sum_{i=1}^m f_i=id_\mathbb{R}\}$. If $(f_1,\dots,f_m)\in\mathbb{A}$ then f_i, $i=1,\dots,m$, are 1-Lipschitz-continuous. This implies $|f_i(X)|\leq|X|+|f_i(0)|$, $i=1,\dots,m$, which leads to

$$\text{if } X\in L^p,\quad (f_1(X),\dots,f_m(X))\in(L^p)^m.$$

We can now use these results to formulate the next proposition which states that the optimization problem on the set $\mathscr{A}(X)$ reduces to the optimization problem on the set $\mathscr{C}(X)$. Furthermore the infimum in (3) is attained for some $Y\in\mathscr{C}(X)$.

Proposition 3.3. *Let $\rho_i^j:L^p\to\mathbb{R}\cup\{\infty\}$, $1\leq j\leq r_i$, $1\leq i\leq m$, $1\leq p\leq\infty$ be law-invariant convex risk measures. Then it follows:*

(a) $\mathscr{G}_S$ takes the form

$$\mathscr{G}_S(X)=\inf_{Z\in\mathscr{C}(X)}\max\{\rho_1^{max}(Z_1),\dots,\rho_m^{max}(Z_m)\} \tag{4}$$
$$=\inf_{(f_1,\dots,f_m)\in\mathbb{A}}\max\{\rho_1^{max}(f_1(X)),\dots,\rho_m^{max}(f_m(X))\}$$

(b) The infimum is attained for some $Y\in\mathscr{C}(X)$ and $Y_i=f_i(X)$ for some $(f_1,\dots,f_m)\in\mathbb{A}$.

Proof. Risk measures on a non-atomic probability space with the required properties are consistent with convex ordering. This follows from Theorem 4.3 in [20]. (a) now follows from the fact that for any $Y\in\mathscr{A}(X)$ there exists $Y'\in\mathscr{C}(X)$ that dominates Y and this is equivalent to the existence of $(f_1,\dots,f_m)\in\mathbb{A}$ such that $Y_i'=f_i(X)$, $i=1,\dots,m$. This first statement follows from Theorem 2 in [21] and the second from Proposition 5.1 in [18] which states that for any allocation $Y\in\mathscr{A}(X)$ there exist

$$(f_1,\dots,f_m)\in\mathbb{A}$$

such that $f_i(X)\succeq_c Y_i$, $i=1,\dots,m$.

Let us now prove (b). We can modify Step 3 of the proof of Theorem 5.2 in [18] to our setting. Suppose $X\in L^p$ is such that $\mathscr{G}_S(X)=\infty$. Then every allocation $(f_1(X),\dots,f_m(X))$,

$$(f_1,\dots,f_m)\in\mathbb{A}$$

is optimal. Now, let $X\in\mathrm{dom}\mathscr{G}_S$ and choose a sequence

$$(f_1^n,\dots,f_m^n)\in\mathbb{A},$$

$n\in\mathbb{N}$, such that

$$\mathscr{G}_S(X)=\lim_{n\to\infty}\max\{\rho_1^{max}(f_1^n(X)),\dots,\rho_m^{max}(f_m^n(X))\}.$$

For $X=0$ we have

$$0=\mathscr{G}_S(0)=\lim_{n\to\infty}\max\{-f_1^n(0),\dots,-f_m^n(0)\}.$$

This implies that for some $K>0$ there exists $N\in\mathbb{N}$ such that $|\max\{-f_1^n(0),\dots,-f_m^n(0)\}|\leq K$ for all $n\geq N$. Since

$$\sum_{i=1}^m f_i^n(0)=0$$

this yields the existence of a constant $L>0$ such that $f_i^n(0)\in[-L,L]$, $i=1,\dots,m$, for all $n\geq N$ and this enables us to choose a subsequence $(f_1^{n_k},\dots,f_m^{n_k})\in\mathbb{A}$, $k\in\mathbb{N}$, such that $f_i^{n_k}(0)\in[-L,L]$, $i=1,\dots,m$, for all $k\in\mathbb{N}$. Now, Lemma 3.1 implies the existence of subsequences $(f_i^{n_{k_l}})_{l\in\mathbb{N}}$, $i=1,\dots,m-1$, and 1-Lipschitz-continuous functions $f_i:\mathbb{R}\to\mathbb{R}$, $i=1,\dots,m-1$, such that

$$f_i(a)=\lim_{l\to\infty}f_i^{n_{k_l}}(a),$$

$i=1,\dots,m-1$, for all $a\in\mathbb{R}$. Then $f_i^{n_{k_l}}(X)$ converges to $f_i(X)$ $\mathbb{P}$-a.s., $i=1,\dots,m-1$ and $f_m^{n_{k_l}}(X)=X-\sum_{i=1}^{m-1}f_i^{n_{k_l}}(X)$ converges to $f_m(X)$ $\mathbb{P}$-a.s. where $f_m:=id_\mathbb{R}-\sum_{i=1}^{m-1}f_i$. Thus

$$(f_1,\dots,f_m)\in\mathbb{A},$$

which means that f_i, $i=1,\dots,m$, is increasing and 1-Lipschitz-continuous. From the 1-Lipschitz-continuity it follows

$$|f_i^{n_{k_l}}(X)|\leq|X|+L,$$

$i=1,\dots,m$, for all $l\in\mathbb{N}$. Since $X\in L^p$ the dominated convergence theorem yields $f_i(X)\in L^p$ and $\|f_i(X)-f_i^{n_{k_l}}(X)\|_p\to0$ for $l\to\infty$, $i=1,\dots,m$. Then we get from the Fatou property

$$\mathscr{G}_S(X)\geq\max\{\rho_1^{max}(f_1(X)),\dots,\rho_m^{max}(f_m(X))\}.$$

On the other hand we get from (4)

$$\mathscr{G}_S(X) \le \max\{\rho_1^{max}(f_1(X)), \dots, \rho_m^{max}(f_m(X))\}.$$

This yields the optimality of the allocation

$$(f_1(X), \dots, f_m(X)) \in \mathscr{C}(X)$$

and we have shown that the infimum is attained in (4).

Note that for L^∞ we can use Lemma 3.2 which states that the set of comonotone allocations of X is convex and compact in L^∞ up to zero sum translations. □

Definition 3.4. *A vector $Y \in \mathscr{C}(X)$ for which the infimum in* (4) *is attained will be called* **optimal risk sharing** *of X among agents with risk measures ρ_i^j for $1 \le j \le r_i$, $1 \le i \le m$.*

The following proposition states properties of $\mathscr{G}_S$ in connection to the properties of the underlying risk measures ρ_i^j for $j = 1, \dots, r_i$, $i = 1, \dots, m$.

Proposition 3.5. *Let $\rho_i^j : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \le j \le r_i$, $1 \le i \le m$, $1 \le p \le \infty$ be law-invariant convex risk measures. Then $\mathscr{G}_S$ is law-invariant, satisfies (R2), (R3) and has the property*

$$\mathscr{G}_S(X - r) \le \mathscr{G}_S(X) + r \quad \text{for any } X \in L^p \text{ and } r \in \mathbb{R}_+. \qquad (5)$$

Proof. Law-invariance follows directly, such that we only have to prove the properties (R2), (R3) and the property (5). Let us start with the property (R2). Let $Z \in \mathscr{C}(X)$ be an optimal risk sharing with respect to risk measures ρ_i^j. The existence of such a Z is guaranteed from Proposition 3.3. Let $X, Y \in L^p$, $Y \ge X$ and let $W \in \mathscr{C}(Y)$ be defined by

$$W_i = \begin{cases} Z_j + (Y - X) & \text{if } i = j, \\ Z_i & \text{if } i \ne j. \end{cases}$$

It follows that $\mathscr{G}_S(Y) \le \mathscr{G}_S(X)$. The other properties follow immediately by using the corresponding properties of the underlying convex risk measures and property (R2) of $\mathscr{G}_S$. □

The next corollary follows immediately since $\mathscr{G}_S$ has property (R4) if the underlying risk measures have the corresponding property.

Corollary 3.6. *Let $\rho_i^j : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \le j \le r_i$, $1 \le i \le m$, $1 \le p \le \infty$ be law-invariant coherent risk measures. Then $\mathscr{G}_S$ is law-invariant, satisfies (R2)-(R4) and has the property*

$$\mathscr{G}_S(X - r) \le \mathscr{G}_S(X) + r \text{ for any } X \in L^p \text{ and } r \in \mathbb{R}_+.$$

Property (5) is called cash-subadditivity. Proposition 3.5 now leads to the following corollary, which is the dual representation of cash-subadditive convex risk measures that was obtained in [11]. Denote with $\mathscr{M}_{1,f}(\Omega, \mathscr{F}, \mathbb{P})$ the positive unit ball of $ba(\Omega, \mathscr{F}, \mathbb{P})$, which means that $\mathscr{M}_{1,f}$ coincides with the set of finitely additive probabilities that are absolutely continuous with respect to $\mathbb{P}$. The subset of countably additive probabilities with q-integrable density is denoted by $\mathscr{M}_{1,q} = \{Q \in \mathscr{M}_1 : \frac{dQ}{d\mathbb{P}} \in L^q\}$ $\forall p \in [1, \infty)$ while $\mathscr{M}_{1,f} = \mathscr{M}_{1,q}$ if $p = \infty$ and where $\mathscr{M}_1$ denotes the subset of $\mathscr{M}_{1,f}$ consisting of all its countably additive elements.

Corollary 3.7. *Let $\rho_i^j : L^p \to \mathbb{R}$, $1 \le j \le r_i$, $1 \le i \le m$, $1 \le p \le \infty$ be law-invariant convex risk measures. Then $\mathscr{G}_S : L^p \to \mathbb{R}$ satisfies (R2), (R3) and property* (5) *and thus obtains the representation*

$$\mathscr{G}_S(X) = \sup_{Q \in \mathscr{M}_{1,q}} G_S(\mathbb{E}_Q(-X), Q) \quad \forall X \in L^p, \qquad (6)$$

where the function G_S for which (6) *holds is given by*

$$G_S(t, Q) = \sup_{c \in [0,1]} (ct - \alpha_S(cQ)), \; \forall (t, Q) \in \mathbb{R} \times \mathscr{M}_{1,q},$$

and where $\alpha_S(\cdot)$ denotes the Fenchel conjugate of $\mathscr{G}_S(-\cdot)$.

Proof. According to Proposition 3.3 the infimum is attained in (4) and Proposition 3.5 holds. Consequently $\mathscr{G}_S$ satisfies (R2), (R3) and property (5). With all these properties $\mathscr{G}_S$ is also quasiconvex and upper semicontinuous. Thus it is evenly quasiconvex and we can apply Theorem A.1 and (1.3) in [11] to obtain the desired result (see also Corollary 4.4 in [22] for $p = \infty$). □

4 Optimality of solutions to the risk sharing problem

In this section we will establish a necessary and sufficient condition for $Y = (Y_1, \dots, Y_m)$ to be an optimal risk sharing of X. In this approach we follow [5], where a similar result was obtained for general deviation measures. For additional information about general deviation measures we refer to [23].

For the next proposition we will need the following result for the subdifferential of the supremum of convex functions. For the proof of this result we refer to [24].

Lemma 4.1. *Let X be a separated locally convex space, let (A, τ) be a separated compact topological space and let $g_\alpha : X \to \mathbb{R} \cup$*

$\{\infty\}$ *be a convex function for every* $\alpha \in A$. *Consider the function* $g := \sup_{\alpha \in A} g_\alpha$ *and* $G(x) := \{\alpha \in A \,|\, g_\alpha(x) = g(x)\}$. *Assume that the mapping* $A \ni \alpha \mapsto g_\alpha(x) \in \mathbb{R} \cup \{\infty\}$ *is upper semicontinuous and* $x_0 \in dom g$ *is such that* g_α *is continuous at* x_0 *for every* $\alpha \in A$. *Then*

$$\partial g(x_0) = \overline{conv}\left\{ \bigcup_{\alpha \in G(x_0)} \partial g_\alpha(x_0) \right\}.$$

In our case we have

$$\mathscr{G}_S(X) = \inf_{Y \in \mathscr{A}(X)} g(Y), \tag{7}$$

where $g(Y) = \max\{g_1(Y), \ldots, g_m(Y)\}$ and the functions g_i are given by $g_i(Y) = \max\{\rho_i^1(Y_i), \ldots, \rho_i^{r_i}(Y_i)\}$, $i = 1, \ldots, m$. If the infimum in (7) is attained in $Y \in \mathscr{A}(X)$ then we have $\mathscr{G}_S(X) = g(Y)$. From Proposition 3.3 we know that the infimum in (7) is attained on $\mathscr{C}(X) \subset \mathscr{A}(X)$ if we use law-invariant convex risk measures ρ_i^j, $j = 1, \ldots, r^i$, $i = 1, \ldots, m$. Thus the application to our setting with finite law-invariant convex risk measures leads to the following simplification of the statement of Lemma 4.1,

$$\begin{aligned} &\partial_Y \max\{g_1(Y), \ldots, g_m(Y)\} \\ &= \mathrm{conv}\left\{ \bigcup_{j: g_j(Y) = \mathscr{G}_S(X)} \partial_Y g_j(Y) \right\} \quad \text{for } Y \in \mathscr{C}(X). \end{aligned} \tag{8}$$

In the following proposition we will provide a characterization of an optimal risk sharing of X.

Proposition 4.2. *Let* $\rho_i^j : L^p \to \mathbb{R}$, $1 \le j \le r_i$, $1 \le i \le m$, $1 \le p \le \infty$ *be law-invariant convex risk measures. Then* $Y \in \mathscr{C}(X)$ *is an optimal risk sharing of* $X \in L^p$ *with respect to risk measures* $\rho_i^1, \ldots, \rho_i^{r_i}$, $1 \le i \le m$ *if and only if the following conditions hold*

(i) $\rho_1^{max}(Y_1) = \cdots = \rho_m^{max}(Y_m)$,

(ii) There exist $\lambda_i > 0$, $i = 1, \ldots, m$ *and some* $Z_0 \in (L^p)^*$ *such that for any* $i = 1, \ldots, m$ *we have*

$$\lambda_i Z_0 \in \partial \rho_i^{max}(Y_i)$$

Proof. Define $g_i(Z) := \max\{\rho_i^1(Z_i), \ldots, \rho_i^{r_i}(Z_i)\}$, for any $Z \in \mathscr{C}(X)$, $i = 1, \ldots, m$. Thus $\mathscr{G}_S$ can be rewritten as

$$\mathscr{G}_S(X) = \inf_{Z \in \mathscr{C}(X)} \max\{g_1(Z), \ldots, g_m(Z)\}.$$

A vector $Y = (Y_1, \ldots, Y_m) \in \mathscr{C}(X)$ minimizes (4) if and only if

$$\begin{aligned} 0 &\in \partial_Y(\max\{\rho_1^{max}(Y_1), \ldots, \rho_m^{max}(Y_m)\}) \\ &= \mathrm{conv}\left\{ \bigcup_{j: g_j(Y) = \mathscr{G}_S(X)} \partial_Y g_j(Y) \right\} \end{aligned} \tag{9}$$

Since the risk measures ρ_i^j on L^p are finite valued they are continuous and subdifferentiable on L^p, $1 \le p \le \infty$. This implies that the functions $\max\{\rho_i^1(\cdot), \ldots, \rho_i^{r_i}(\cdot)\}$ are continuous and subdifferentiable and the equation in (9) is true, see Lemma 4.1 and Proposition 3.1 in [15]. Since $Y_m = X - \sum_{j=1}^{m-1} Y_j$ the subdifferential

$$\begin{aligned} &\partial_Y(\max\{\rho_1^{max}(Y_1), \ldots, \rho_m^{max}(Y_m)\}) \\ &= \partial_Y(\max\{g_1(Y), \ldots, g_m(Y)\}) \end{aligned}$$

is taken with respect to $Y_1, \ldots, Y_{m-1}$, such that (see also the proof of Proposition 3.14 in [10])

$$\partial_{(Y_1, \ldots, Y_{m-1})} g_k(Y) = (0, \ldots, 0, \partial_{Y_k} \rho_k^{max}(Y_k), 0, \ldots, 0) \tag{10}$$

$$\partial_{(Y_1, \ldots, Y_{m-1})} g_m(Y) = (-\partial_{Y_m} \rho_m^{max}(Y_m), \ldots, -\partial_{Y_m} \rho_m^{max}(Y_m)). \tag{11}$$

for $k = 1, \ldots, m-1$,

If (i) holds, i.e. $g_1(Y) = \cdots = g_m(Y)$, then the union in (9) is taken over all $j = 1, \ldots, m$ and thus (ii) implies (9) and it follows that $Y \in \mathscr{C}(X)$ is an optimal risk sharing.

Now, let $Y \in \mathscr{C}(X)$ be an optimal risk sharing. This is equivalent to (9). Assume that $g_j(Y) < \mathscr{G}_S(X)$ for some $j \in \{1, \ldots, m\}$. With (9)-(11) this implies $0 \in \partial_{Y_i} g_i(Y)$ for some $i \ne j$. Now, property (R1) immediately leads to $g_j(Y) = g_i(Y) = \mathscr{G}_S(X)$, which is a contradiction to our assumption $g_j(Y) < \mathscr{G}_S(X)$. Thus (i) follows. With (i) it follows from (9) that $\lambda_j Z_j + \lambda_m(-Z_m) = 0$ for some $Z_j \in \partial_{Y_j} g_j(Y)$ and $Z_m \in \partial_{Y_m} g_m(Y)$, where $\lambda_j \ge 0$, $j = 1, \ldots, m$, and $\sum_{j=1}^{m} \lambda_j = 1$, see equations (10) and (11). Now, from property (R1) of $\rho_m^{max}(\cdot)$ it follows that $0 \notin \partial_{Y_m} \rho_m^{max}(Y_m)$ and $Z_m \ne 0$. This implies $\lambda_j > 0$ for $j = 1, \ldots, m$. Thus condition (ii) holds for $Z_0 = Z_m$. □

The next corollary is a direct consequence of Proposition 4.2 and Remark 2.1 which states that the subgradients of law-invariant convex risk measures on L^∞ that satisfy the Lebesgue property are from L^1.

Corollary 4.3. *Let* $\rho_i^j : L^\infty \to \mathbb{R}$, $1 \le j \le r_i$, $1 \le i \le m$, *be law-invariant convex risk measures that satisfy the Lebesgue property. Then* $Y \in \mathscr{C}(X)$ *is an optimal risk sharing of* $X \in L^\infty$ *with* *respect to risk measures* $\rho_i^1, \ldots, \rho_i^{r_i}$, $i = 1, \ldots, m$ *if and only if the following conditions hold*

(i) $\rho_1^{max}(Y_1) = \cdots = \rho_m^{max}(Y_m)$,

(ii) There exist $\lambda_i > 0$, $i = 1, \ldots, m$ *and some* $Z_0 \in L^1$ *such that for any* $i = 1, \ldots, m$ *we have*

$$\lambda_i Z_0 \in \partial \rho_i^{max}(Y_i)$$

Remark 4.4. *Note that the following results are similar to the contributions in [5] and [10]. In those papers the authors have studied the coalitional risk measure in conjunction with general deviation measures* $\mathscr{D} : L^2 \to [0, \infty]$ *instead of convex risk measures. Although there is one-to-one correspondence between general deviation measures and law-invariant coherent risk measures on a non-atomic probability space there is no such relation to convex risk measures. Furthermore, the techniques used in those contributions rely heavily on the fact that general deviation measures satisfy* $\mathscr{D}(X) > 0$ *for all non-constant X, a property which is not reasonable in our approach to risk measurement with convex risk measures. Instead, we will rely on the translation property (R1) of convex risk measures (see* (12)*) to obtain the desired results.*

Now, fix a random vector $Y^* \in \mathscr{A}(X)$ and define

$$t_i = \rho_i^{max}(Y_i^*),$$

$i = 1, \ldots, m$. Furthermore define

$$\hat{\rho}_i^j(Z) = \rho_i^j(Z + t_i), \quad 1 \leq j \leq r_i, 1 \leq i \leq m. \tag{12}$$

Note that these translated risk measures $\hat{\rho}_i^j$ inherit the translation property, the convexity and the monotonicity from the underlying risk measures ρ_i^j. But while ρ_i^j satisfy $\rho_i^j(0) = 0$ the translated versions satisfy $\hat{\rho}_i^j(0) = -t_i$. Nevertheless, the result of Proposition 4.2 also holds for $\hat{\rho}_i^j$. We only have to suitably replace X' at the respective positions in the proof. This leads to the next result, that characterizes optimal risk sharing in terms of the functional

$$\hat{\mathscr{G}}_S(X) = \inf_{Z \in \mathscr{A}(X)} \max\{\hat{\rho}_1^{max}(Z_1), \ldots, \hat{\rho}_m^{max}(Z_m)\}, \tag{13}$$

where we know from Proposition 3.3 that the infimum can be formed over $\mathscr{C}(X)$ instead of $\mathscr{A}(X)$.

Now, the next proposition follows immediately.

Proposition 4.5. *Let* $\rho_i^j : L^p \to \mathbb{R}$, $1 \leq j \leq r_i$, $1 \leq i \leq m$, $1 \leq p \leq \infty$ *be law-invariant convex risk measures. Then* $Y^* \in \mathscr{C}(X)$ *is an optimal risk sharing with respect to risk measures* $\hat{\rho}_i^j$, $1 \leq j \leq r_i$, $1 \leq i \leq m$, *if and only if* $\hat{\mathscr{G}}_S(X) = 0$.

Now, let us establish a connection to Pareto optimality. Suppose that each agent is equipped with a utility function $U_i : L^p \to (-\infty, \infty]$, which in our setting will have a specific dependence on the underlying risk measures, namely

$$U_i(X) = V_i(\rho_i^{max}(X)), \tag{14}$$

$i = 1, \ldots, m$, $X \in L^p$. We will assume V_i, $i = 1, \ldots, m$, to be strictly decreasing. The optimization problem that arises from the description of the problem the agents face is

$$\max_{Y \in \mathscr{A}(X)} \{U_1(Y_1), \ldots, U_m(Y_m)\},$$

which in our multiple risk setting becomes

$$\max_{Y \in \mathscr{A}(X)} \{V_1(\rho_1^{max}(Y_1)), \ldots, V_m(\rho_m^{max}(Y_m))\}, \tag{15}$$

$i = 1, \ldots, m$. We will approach the problem by Pareto optimality. $Y = (Y_1, \ldots, Y_m) \in \mathscr{A}(X)$ is called Pareto-optimal division in (15) if there is no $Z \in \mathscr{A}(X)$ such that

$$\rho_i^{\max}(Z_i) \leq \rho_i^{\max}(Y_i) \quad i = 1, \ldots, m, \tag{16}$$

with at least one inequality being strict. Note that Pareto optimality, as formulated in (16), does not depend on the functions $V_1, \ldots, V_m$ from (15) such that a Pareto optimal solution is Pareto optimal for any functions V_i, $i = 1, \ldots, m$ that are strictly decreasing.

To show that there are incentives in the sense of (16) to form a coalitional portfolio instead of investing individually we provide the following example.

Example 4.6. *Suppose that each agent has to take into account two risk measures. Agent 1 takes into account* $\rho_1^1(X) = CVaR_{0.05}(X)$ *and* $\rho_1^2(X) = \mathbb{E}[-X] + \sigma(X)$*, whereas agent 2 takes into account* $\rho_2^1(X) = CVaR_{0.05}(X)$ *and* $\rho_2^2(X) = ent_3(X)$*. This means that, according to our considerations, agent 1 uses*

$$\max\{CVaR_{0.05}(X), \mathbb{E}[-X] + \sigma(X)\}$$

and agent 2 uses $\max\{CVaR_{0.05}(X), ent_3(X)\}$*, where CVaR denotes Conditional-Value-at-Risk and* ent_γ *denotes the entropic* *risk measure with parameter* γ. *We assume that agent i invests capital* $x_i \in (0,1)$ *in a risky asset* $Z_i \sim \mathcal{N}(1,2^2)$, *whereas* Z_i *are i.i.d. random variables. This implies that both agents could form a coalitional portfolio* $X_S = X_1 + X_2 = x_1 Z_1 + x_2 Z_2$.

Now, define $Y_1 = (x_1/(x_1+x_2))X_S$ *and* $Y_2 = (x_2/(x_1+x_2))X_S$. *Then* $(Y_1, Y_2) \in \mathscr{C}(X_S)$. *Furthermore we have*

$$\max\{CVaR_{0.05}(Y_1), \mathbb{E}[-Y_1] + \sigma(Y_1)\} < \max\{CVaR_{0.05}(X_1), \mathbb{E}[-X_1] + \sigma(X_1)\}$$

$$\max\{CVaR_{0.05}(Y_2), ent_3(Y_2)\} < \max\{CVaR_{0.05}(X_2), ent_3(X_2)\},$$

which means that both agents can reduce their risk by forming a cooperative portfolio X_S *and investing according to* (Y_1, Y_2).

The next proposition connects optimal risk sharing in terms of our coalitional risk measure to Pareto optimal solutions. This result follows immediately.

Proposition 4.7. *Let* $\rho_i^j : L^p \to \mathbb{R}$, $1 \le p \le \infty$ *be a law-invariant convex risk measure for any* $1 \le j \le r_i$, $1 \le i \le m$. *A vector* $Y^* \in \mathscr{C}(X)$ *with* $t_i = \rho_i^{max}(Y_i^*)$, $i = 1, \dots, m$, *is a Pareto optimal solution to* (15) *if and only if* $\hat{\mathscr{G}}_S(X) = 0$, *where* $\hat{\rho}_i^j$, $1 \le j \le r_i$, $1 \le i \le m$, *are defined in* (12).

Now from the previous proposition and the characterization result in Proposition 4.2 we can state the following theorem. It characterizes solutions to problem (15) in terms of the subdifferential of the coalitional risk measure $\hat{\mathscr{G}}_S$ and is the main result of this work.

Theorem 4.8. *Let* $\rho_i^j : L^p \to \mathbb{R}$, $1 \le j \le r_i$, $1 \le i \le m$, $1 \le p \le \infty$ *be law-invariant convex risk measures. A random vector* $Y^* \in \mathscr{C}(X)$ *with with* $t_i = \rho_i^{max}(Y_i^*)$, $i = 1, \dots, m$, *is a Pareto optimal solution to* (15) *if and only if there exist* $\lambda_i > 0$, $i = 1, \dots, m$, *and* $Z_0 \in L^p(\Omega, \mathscr{F}, \mathbb{P})^*$ *such that*

$$\lambda_i Z_0 \in \partial \rho_i^{max}(Y_i^*)$$

for any $i = 1, \dots, m$.

Proof. We know from Proposition 4.7 that $Y^* \in \mathscr{C}(X)$ is a Pareto optimal solution to (15) if and only if $\hat{\mathscr{G}}_S(X) = 0$. According to Proposition 4.5 this is the case if and only if $Y^* \in \mathscr{C}(X)$ is an optimal risk sharing with respect to risk measures $\hat{\rho}_i^j$, $j = 1, \dots, r_i$, $i = 1, \dots, m$. Now the result follows from the characterization result of optimal risk sharing in Proposition 4.2 and the definition of the subdifferential. □

As before the next corollary is a direct consequence of Theorem 4.8 and Remark 2.1 which states that the subgradients of law-invariant convex risk measures on L^∞ that satisfy the Lebesgue property are from L^1.

Corollary 4.9. *Let* $\rho_i^j : L^\infty \to \mathbb{R}$, $1 \le j \le r_i$, $1 \le i \le m$ *be law-invariant convex risk measures that satisfy the Lebesgue property. A random vector* $Y^* \in \mathscr{C}(X)$ *with* $t_i = \rho_i^{max}(Y_i^*)$, $i = 1, \dots, m$, *is a Pareto optimal solution to* (15) *if and only if there exist* $\lambda_i > 0$, $i = 1, \dots, m$, *and* $Z_0 \in L^1(\Omega, \mathscr{F}, \mathbb{P})$ *such that*

$$\lambda_i Z_0 \in \partial \rho_i^{max}(Y_i^*)$$

for any $i = 1, \dots, m$.

We conclude our paper by applying Theorem 4.8 to Example 4.6.

Example 4.10 (Application of our main result to Example 4.6)**.** *Note that with the assumptions of Example 4.6 we have*

$$\max\{CVaR_{0.05}(Y_1), \mathbb{E}[-Y_1] + \sigma(Y_1)\} = CVaR_{0.05}(Y_1)$$

and $\max\{CVaR_{0.05}(Y_2), ent_3(Y_2)\} = CVaR_{0.05}(Y_2)$. *This implies*

$$\partial CVaR_{0.05}(Y_1) \subseteq \partial \max\{CVaR_{0.05}(Y_1), \mathbb{E}[-Y_1] + \sigma(Y_1)\}$$

and $\partial CVaR_{0.05}(Y_2) \subseteq \partial \max\{CVaR_{0.05}(Y_2), ent_3(Y_2)\}$. *Furthermore, note that the subdifferential of* $CVaR_\beta$ *at X is given by*

$$\partial CVaR_\beta(X) = \left\{ Z \in L^q : \mathbb{E}[Z] = -1, \left\{ \begin{matrix} Z(\omega) = -1/\beta & X(\omega) < -VaR_\beta(X) \\ Z(\omega) \in [-1/\beta, 0] & X(\omega) = -VaR_\beta(X) \\ Z(\omega) = 0 & X(\omega) > -VaR_\beta(X) \end{matrix} \right\} \right\}$$

This implies, due to the positive homogeneity of VaR, and the positivity of $x_i/(x_1+x_2)$, $i = 1,2$,

$$\partial CVaR_{0.05}(Y_1) = \partial CVaR_{0.05}\left(\frac{x_1}{x_1+x_2} X_S\right) = \partial CVaR_{0.05}(X_S) = \partial CVaR_{0.05}\left(\frac{x_2}{x_1+x_2} X_S\right) = \partial CVaR_{0.05}(Y_2)$$

Now, our theorem immediately implies that $(Y_1, Y_2) \in \mathscr{C}(X_S)$ *is a Pareto optimal solution to* (15).

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