Ordered Contribution Allocations: Theoretical Properties and Applications

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Abstract

We introduce two general cost allocation methods that are based on the players' ordered contributions to the total cost. The resulting allocations always are efficient in a game theoretic sense, and they are symmetric if the underlying order is symmetric with respect to the cost function. While the first of our methods is easily implementable for a large number of players, the second one always satisfies the dummy player property but, depending on the underlying order, can be computationally expensive. Our main application is a problem where the risk of an aggregate financial position has to be distributed among its constituent parts. In contrast with most existing methods our approach also works if the underlying risk measure has no particular properties such as coherency, convexity or differentiability. Therefore, it can for instance, be used in conjunction with Value-at-Risk, a widely used risk measure with none of the above properties. Moreover, it gives every player incentives to behave in a way that keeps the risk of the aggregate position low. As a second application we propose a nonlinear tax scheme for emissions of a pollutant such as CO_2 that is able to enforce strict limits on total emissions.

Keywords Cost allocation, ordered contributions, risk allocation, emission taxes.

1 Introduction

There are various contexts in which a cost generated by an entity such as a firm or industry has to be allocated among its constituent parts. The cost could, for instance, be a monetary expense, the risk of a financial position or environmental damage caused by industrial activities. In many cases the cost accrued by the entity as a whole is different from the sum of the costs corresponding to the parts when viewed as individual units, and the cost allocation problem naturally arises in different applications. We use the language of cooperative game theory and call the parts of our entity "players". We assume there is a finite set of them and denote it by $N = \{1, \dots, n\}$. We let 2^N be the set of all possible subsets of N and consider a function $c : 2^N \to \mathbb{R}$ such that $c(\emptyset) = 0$. For every subset $S \subset N$, the number $c(S)$ is meant to be a cost caused by the group of players in S. Formally, an allocation is a vector $k = (k_1, \dots, k_n) \in \mathbb{R}^n$ specifying how much of the total cost $c(N)$ is allocated to each player. Some properties an

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allocation can have are:

Efficiency $c(N) = \sum_{i=1}^{n} k_i$.
Symmetry $k_i = k_j$ if $c(\{i\} \cup S) = c(\{j\} \cup S)$ for all $S \subset N \setminus \{i, j\}$
Dummy player property $k_i = c(i)$ if $c(\{i\} \cup S) = c(i) + c(S)$ for every subset $S \subset N \setminus \{i\}$.

(In the risk allocation literature, efficiency is usually called full allocation.)

We propose two general cost allocation methods that are based on the cost increments if the players are added to each other according to a certain order. However, we allow for several of them to be grouped together and added at the same time. Our two methods differ in how they distribute the cost among players in the same group. The allocations resulting from both our methods always have the efficiency property, and they are symmetric if the underlying order is symmetric with respect to the cost function c (as defined in Definition 2.2 below). The advantage of the first method is that it is easily implementable for a large number of players n if the cost $c(S)$ can efficiently be computed for every subset $S \subset N$. But in general it does not have the dummy player property. The second method always has the dummy player property but might become too complex to calculate exactly for large n.

Our motivating example is a risk allocation problem of the following form: Consider n uncertain payoffs modeled by random variables $X_1, \ldots, X_n$ and a risk measure ρ. The question is how to distribute the risk $\rho(\sum_{i=1}^{n} X_i)$ of the aggregate position among its constituents. This problem arises, for example, in portfolio management, internal risk management, performance measurement, employee compensation schemes or financial regulation. Several approaches have been proposed in the finance and insurance mathematics literature; see Section 3 below. But most of them require the risk measure ρ to be coherent, convex or differentiable. The advantage of our approach is that it does not need ρ to have any specific properties and always leads to an efficient allocation. In particular, it can be employed in conjunction with Value-at-Risk, which is the most widely used risk measure in the financial industry but does not have any of the above properties. Moreover, it provides strong incentives to the player to behave in a way that keeps overall risk low. In a second example we propose a nonlinear tax scheme for emissions of pollutants such as CO_2. Our tax scheme determines individual taxes on the basis of everybody's emissions and is therefore better able to control total emissions than traditional tax rules.

The structure of the paper is as follows: In Section 2 we introduce our two cost allocation methods and study their theoretical properties. In Section 3 we apply them to the problem of distributing the risk of an aggregate financial position and compare them to other risk allocation principles that have been proposed in the literature. In Section 4 we introduce a nonlinear tax scheme for emissions of a pollutant. In Section 5 the computational complexity of the two methods is discussed.

2 Ordered contribution allocations

Let $\preceq$ be a total preorder on N, that is, a binary relation with the following two properties:

Transitivity If $i \preceq j$ and $j \preceq l$, then $i \preceq l$
Totality $i \preceq j$ or $i \succeq j$ for all $i, j \in N$.

Note that totality implies reflexivity: $i \preceq i$ for all $i \in N$. In our examples below, the preorder is generated by the cost function c, but it does not have to be.

Every preorder on N is fully characterized by the vector $(\sigma_1, \ldots, \sigma_m)$ of its level sets, which

forms an ordered partition of N:

$$N = \bigcup_{i=1}^{m} \sigma_i \quad \text{and} \quad \sigma_i \cap \sigma_j = \emptyset \quad \text{for } i \neq j.$$

On the other hand, every ordered partition of N induces a total preorder on N through

$$i \preceq j \Leftrightarrow i \in \sigma_l,\ j \in \sigma_{l'} \quad \text{for } l \leq l'.$$

So total preorders are in one-to-one correspondence with ordered partitions.

Our first allocation method is as follows:

Definition 2.1 *The ordered contribution allocation corresponding to an ordered partition $(\sigma_1, \ldots, \sigma_m)$ of N is defined by*

$$k_i := \frac{c(\Sigma_l) - c(\Sigma_{l-1})}{|\sigma_l|} \quad \text{for } i \in \sigma_l, \qquad \text{where } \Sigma_l := \sigma_1 \cup \cdots \cup \sigma_l \text{ and } \Sigma_0 := \emptyset. \tag{2.1}$$

Note that for a given ordered partition $(\sigma_1, \ldots, \sigma_m)$, the cost function c only needs to be evaluated $m \leq n$ times to compute (2.1). But usually, the partition also needs to be calculated. Concrete examples of ordered partitions are discussed in Subsections 2.1–2.3 below.

Definition 2.2 *We call a total preorder or the corresponding ordered partition symmetric with respect to c if $i \sim j$ whenever $c(\{i\} \cup S) = c(\{j\} \cup S)$ for all $S \subset N \setminus \{i, j\}$.*

It is clear that allocation (2.1) has the following properties:

Proposition 2.3 *The ordered contribution allocation* (2.1) *is always efficient, and it is symmetric if the underlying ordered partition is symmetric with respect to c. Moreover, it has the dummy player property if every level set σ_l consists of only one player.*

In Definition 2.4 below we present a slightly different allocation principle which always has the dummy player property. Instead of dividing the cost contribution of a group σ_l by the number of its members, it calculates the average of the individual contributions when the members of a group are added up one after the other according to all possible permutations of the group.

For a given ordered partition $\sigma = (\sigma_1, \ldots, \sigma_m)$ of N we denote by $\Pi(\sigma)$ the set of all permutations of N that leave every subset σ_l invariant.

Definition 2.4 *The average ordered contribution allocation corresponding to an ordered partition $(\sigma_1, \ldots, \sigma_m)$ of N is given by*

$$k_i := \frac{1}{|\Pi(\sigma)|} \sum_{\pi \in \Pi(\sigma)} c\left\{\pi(1), \ldots, \pi(i)\right\} - c\left\{\pi(1), \ldots \pi(i-1)\right\}, \tag{2.2}$$

where $\{\pi(1), \ldots, \pi(i-1)\}$ is meant to be $\emptyset$ for $\pi(i) = 1$.

It can easily be verified that the following holds:

Proposition 2.5 *The average ordered contribution allocation* (2.2) *always satisfies efficiency and the dummy player property. Moreover, it is symmetric if the underlying ordered partition is symmetric with respect to the cost function c.*

In the special case where every level set σ_l contains only one player, (2.2) reduces to (2.1). But in the other extreme case $\sigma_1 = N$, (2.2) becomes the Shapley value (see Shapley, 1953) and requires the evaluation of $c(S)$ for every possible subset S of N (see Section 5 below). For a general ordered partition, allocation (2.2) is a special instance of the Owen value (see Owen (1977) or Hart and Kurz (1983)).

In the following three subsections we focus on three special examples of ordered partitions.

2.1 Minimal increments ordering

The most natural ordered partition in conjunction with the allocation principles (2.1) and (2.2) is given by

$$\sigma_l := \arg\min_{i \in \Sigma_{l-1}^c} c(\{i\} \cup \Sigma_{l-1}), \quad \text{where } \Sigma_0 := \emptyset. \tag{2.3}$$

Since it is symmetric with respect to c, the corresponding allocations (2.1) and (2.2) are both symmetric. Note that to calculate (2.3), c needs to be evaluated at most $n(n+1)/2$ times. Moreover, one has the following:

Proposition 2.6 *The ordered contribution allocation* (2.1) *corresponding to the ordered partition* (2.3) *has the dummy player property if each σ_l contains at most two players.*

Proof. Assume there exists $i \in N$ such that

$$c(\{i\} \cup S) = c(i) + c(S) \quad \text{for every subset } S \subset N \setminus \{i\}. \tag{2.4}$$

If $\sigma_l = \{i\}$ for some l, it is clear that $k_i = c(i)$. Otherwise, there exist j and l such that $\sigma_l = \{i, j\}$, and one has

$$\begin{aligned} k_i &= \frac{1}{2}\left(c(\{i,j\} \cup \Sigma_{l-1}) - c(\Sigma_{l-1})\right) \\ &= \frac{1}{2}\left(c(i) + c(\{j\} \cup \Sigma_{l-1}) - c(\Sigma_{l-1})\right) \\ &= \frac{1}{2}\left(2c(i) + c(\Sigma_{l-1}) - c(\Sigma_{l-1})\right) = c(i). \end{aligned}$$

The second equality follows from (2.4) and the third from the fact that i and j belong to the same level set σ_l. □

2.2 Last contribution ordering

Another possible ordering is based on the contribution of each player to the total cost when viewed as the last contributor:

$$\sigma_l := \arg\min_{i \in \Sigma_{l-1}^c} (c(N) - c(N \setminus \{i\})) = \arg\max_{i \in \Sigma_{l-1}^c} c(N \setminus \{i\}), \quad \text{where } \Sigma_0 := \emptyset. \tag{2.5}$$

Since this ordering is symmetric with respect to c, the corresponding allocations (2.1) and (2.2) become symmetric. The last contribution ordering (2.5) is related to the with-and-without allocation proposed in Merton and Perold (1993) or Matten (1996), where the allocation is specified directly by formula (2.5). However, here only the ordered partition is determined by (2.5). To determine this ordered partition, the cost function c has to be evaluated n times.

2.3 Individual cost ordering

A third way of ordering the players is according to their individual costs:

$$\sigma_l := \arg\min_{i \in \Sigma_{l-1}^c} c(i), \quad \text{where } \Sigma_0 := 0. \tag{2.6}$$

Again it is symmetric with respect to c, and the corresponding allocations (2.1) and (2.2) become symmetric. Note that while the ordering is just based on the individual costs, the allocations (2.1) and (2.2) still depend on each player's contribution to the total cost. As before, c needs to be evaluated n times for the computation of (2.6).

3 Allocating the risk of an aggregate financial position

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and denote by L^0 the space of all random variables, where two of them are identified if they agree $\mathbb{P}$-almost surely. Let $\mathcal{X}$ be a subspace of L^0 containing the constants and consider n financial positions with random payoffs $X_1, \ldots, X_n \in \mathcal{X}$. We think of them as future profits of n units of a firm which is concerned with the total risk $\rho(X_1 + \cdots + X_n)$ for some risk measure $\rho : \mathcal{X} \to \mathbb{R}$. Let us assume that, as a basis for performance measurement or internal risk management, the firm wants to allocate the total risk to the units. We define the cost function $c : 2^N \to \mathbb{R}$ associated with this problem by

$$c(\emptyset) := 0 \quad \text{and} \quad c(S) := \rho\left(\sum_{i \in S} X_i\right) \quad \text{for non-empty subsets } S \text{ of } N.$$

We do not require ρ to have any particular properties. But recently risk measures with some or all of the features below have become popular. ρ is said to be coherent if it has all four of them and convex if it has the first three.

Monotonicity $\rho(X) \geq \rho(Y)$ for $X \leq Y$
Translation property $\rho(X + m) = \rho(X) - m$ for all $X \in \mathcal{X}$ and $m \in \mathbb{R}$.
Convexity $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for all $X, Y \in \mathcal{X}$ and $\lambda \in (0,1)$
Positive homogeneity $\rho(\lambda X) = \lambda\rho(X)$ for all $X \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+$.

The translation property above is stated for the case where X is understood as a discounted future profit. In undiscounted terms it reads as $\rho(X + mr) = \rho(X) - m$, where r is the total rate of return on a riskless investment; see Artzner el al. (1999), Denault (2001) or Föllmer and Schied (2004). We point out that under positive homogeneity, convexity is equivalent to the following property:

Subadditivity $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{X}$.

Also, note that if ρ has the convexity property, then the directional derivative

$$\rho'(X; Z) := \lim_{\varepsilon \downarrow 0} \frac{\rho(X + \varepsilon Z) - \rho(X)}{\varepsilon}$$

exists for all $X, Z \in \mathcal{X}$. But if ρ is not Gâteaux-differentiable, it can happen that

$$\rho'(X; Z) \neq -\rho'(X; -Z) \quad \text{for certain } X, Z \in \mathcal{X},$$

that is, the mapping $\varepsilon \mapsto \rho(X + \varepsilon Z)$ is not differentiable at $\varepsilon = 0$. In fact, it is typical for a coherent risk measure ρ that $\rho(Z) + \rho(-Z) > 0$ for some $Z \in \mathcal{X}$. Due to positive homogeneity, this implies $\rho'(0; Z) > -\rho'(0; -Z)$, and consequently, ρ cannot be Gâteaux-differentiable at 0.

Examples 3.1

1. The *Value-at-Risk* at level $\alpha \in (0,1)$, given by $\mathrm{VaR}_\alpha(X) = \inf\{m \in \mathbb{R} : \mathbb{P}[X + m < 0] \leq \alpha\}$, is real-valued on L^0. It is one of the most widely used risk measures in practice. It has the monotonicity, translation and positive homogeneity properties above, but it fails to be convex and it does not have directional derivatives $\rho'(X; Z)$ for all $X, Z \in L^0$.

2. *Average-Value-at-Risk* at level $\alpha \in (0,1)$ given by $\mathrm{AVaR}_\alpha(X) = \alpha^{-1} \int_0^\alpha \mathrm{VaR}_y(X) dy$ is a coherent risk measure on L^1.

3. The *entropic risk measure* $\mathrm{Ent}_\gamma(X) = \gamma^{-1} \log \mathbb{E}\left[\exp(-\gamma X)\right]$ with parameter $\gamma > 0$ is equal to the negative of the certainty equivalent of expected exponential utility. It is a convex risk measure on the space of random variables with all exponential moments which is not positively homogeneous.

4. The *transformed norm risk measure*

$$T^p_{\alpha,\beta}(X) = \min_{s \in \mathbb{R}} \left(\frac{1}{\alpha} \left\| (s - X)^+ \right\|_p^\beta - s \right)$$

with parameters $\alpha \in (0,1)$ and $\beta, p \geq 1$ is a convex risk measure on L^p, which is coherent if and only if $\beta = 1$ and coincides with AVaR_α if $\beta = p = 1$; see Cheridito and Li (2009).

5. Let $\varphi : [0,1] \to [0,1]$ be an increasing function satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$. The *Choquet integral* of X with respect to the distorted probability $\varphi \circ \mathbb{P}$ is defined by

$$\mathbb{E}_\varphi[X] = \int_{-\infty}^0 (\varphi \circ \mathbb{P}[X > t] - 1) dt + \int_0^\infty \varphi \circ \mathbb{P}[X > t] dt.$$

If φ is concave, then $\mathbb{E}_\varphi[-X]$ defines a coherent risk measure on L^∞; see Delbaen (2002).

6. Let $\gamma > 0$ and $\varphi : [0,1] \to [0,1]$ be an increasing concave function such that $\varphi(0) = 0$ and $\varphi(1) = 1$. The *distorted entropic risk measure*

$$\mathrm{Ent}_{\gamma,\varphi}(X) = \frac{1}{\gamma} \log \mathbb{E}_\varphi[\exp(-\gamma X)]$$

is a convex risk measure on L^∞, which has been studied in Tsanakas (2009).

The advantage of the ordered contribution allocation (2.1) and the average ordered contribution allocation (2.2) in a risk allocation context is their simplicity as well as the fact that they do not require the risk measure ρ to have any particular properties. For instance, they work in conjunction with Value-at-Risk. Both allocations, (2.1) and (2.2), are efficient and symmetric if, for instance, they are based on the minimal increments ordering (2.3). Moreover, they give the units incentives to take positions so that the risk of the aggregate exposure $X_1 + \cdots + X_n$ stays low. Provided that $\rho\left(\sum_{i \in S} X_i\right)$ can efficiently be calculated for all subsets S of N, (2.1) is easy to implement for large n. If in addition, the risk measure ρ has the translation property and satisfies $\rho(0) = 0$, then (2.1) fulfils the following weaker version of the dummy player property, which was introduced by Denault (2001):

Riskless allocation $k_i = \rho(X_i)$ for all i such that X_i is a constant.

The average ordered contribution allocation (2.2) on the other hand, always has the dummy player property. But depending on the underlying order, its computational cost can grow exponentially in n (a more detailed discussion of the computational complexity of the allocation methods (2.1) and (2.2) is given in Section 5 below).

Other simple risk allocation rules that can be used with an arbitrary risk measure ρ are the proportional allocation

$$k_i = \frac{\rho(X_i)}{\sum_{j=1}^n \rho(X_j)}\rho(X) \tag{3.1}$$

proposed by Urban et al. (2003) and the with-and-without allocation

$$k_i = \rho(X) - \rho(X - X_i), \tag{3.2}$$

see Merton and Perold (1993) or Matten (1996). The proportional allocation is efficient, but it needs $\sum_{j=1}^n \rho(X_j)$ to be different from zero. In general, the with-and-without allocation does not have the efficiency property. But if $\sum_{j=1}^n \rho(X) - \rho(X - X_j)$ is non-zero, one can normalize it to get the efficient allocation

$$k_i = \frac{\rho(X) - \rho(X - X_i)}{\sum_{j=1}^n \rho(X) - \rho(X - X_j)}\rho(X). \tag{3.3}$$

The marginal version of (3.2) is

$$k_i = \lim_{\varepsilon \downarrow 0} \frac{\rho(X) - \rho(X - \varepsilon X_i)}{\varepsilon} = -\rho'(X; -X_i). \tag{3.4}$$

If

$$\rho'(X; X_i) = -\rho'(X; -X_i) \quad \text{for all } i,$$

it coincides with

$$k_i = \lim_{\varepsilon \downarrow 0} \frac{\rho(X + \varepsilon X_i) - \rho(X)}{\varepsilon} = \rho'(X; X_i) \tag{3.5}$$

and is sometimes called Euler allocation or gradient allocation, see, for instance, Patrik et al. (1999) or Tasche (1999). (3.4) and (3.5) require ρ to have directional derivatives, and like (3.2), in general they do not satisfy efficiency. However, if ρ is positively homogeneous and Gâteaux-differentiable at X, then the Euler allocation is efficient; see Tasche (1999), Denault (2001), Fischer (2003) or McNeil et al. (2005). Value-at-Risk is positively homogenous but not Gâteaux-differentiable. So the Euler allocation can only be used with Value-at-risk if one makes regularity assumptions on the joint distribution of $(X_1, \ldots, X_n)$; see, for instance, Garman (1997), Gourieroux and Scaillet (2000), Tasche (2000) or McNeil et al. (2005). The use of the Euler allocation with Average-Value-at-Risk is discussed in Kalkbrener et al. (2004) and with the more general class of spectral risk measures in Overbeck and Sokolova (2009). It follows from the results in Kalkbrener (2005) that if ρ is positively homogeneous, subadditive and $\rho'(X; Z) = -\rho'(X; -Z)$ for all $Z \in \mathcal{X}$, then the Euler allocation is efficient and in addition satisfies

No undercut $\sum_{i \in S} k_i \leq \rho\left(\sum_{i \in S} X_i\right)$ for every subset S of N.

Efficient allocations with the no undercut property have also been studied in situations where ρ is not differentiable; see for instance, Delbaen (2000), Denault (2001), Kalkbrener (2005), Cherny and Orlov (2007) and Furman and Zitikis (2008). However, the no undercut property is related to subadditivity of ρ, and it cannot be hoped that an efficient allocation with the no undercut property exists if ρ is not subadditive; see Buch and Dorfleitner (2008). For Gâteaux-differentiable risk measures satisfying $\rho(0) = 0$, Tsanakas (2009) proposes to use the Aumann–Shapley value (see Aumann and Shapley (1974))

$$k_i = \int_0^1 \rho'(\lambda X; X_i) d\lambda \tag{3.6}$$

as an allocation rule. It is shown in Cheridito and Li (2008) that $T^p_{\alpha,\beta}$ is Gâteaux-differentiable if $\beta, p > 1$ and in Tsanakas (2009) that $\mathrm{Ent}_{\gamma,\varphi}$ is Gâteaux-differentiable if φ is differentiable and the quantile function F_X^{-1} of the aggregate position X is strictly increasing.

4 Tax schemes for pollution emission

We here understand the players as companies that emit a pollutant such as CO_2 and denote by $x_i \in \mathbb{R}$ the amount of pollutant emitted by company i. Traditional tax schemes charge each company a tax of the form $f(x_i)$ for some function $f : \mathbb{R}_+ \to \mathbb{R}_+$. In the simplest case, the function is linear: $f(x_i) = ax_i$ for some fixed constant $a > 0$. However, a country or an economic union like the EU might want to increase the marginal rate depending on the aggregate emission or even set a limit on the total amount that can be emitted. One way of achieving this is a cap and trade scheme; see e.g., Wicke (2005) for an overview. Alternatively, one can introduce taxes that depend on the whole vector $(x_1, \ldots, x_n)$; for a general comparison of CO_2 emissions trading schemes and taxes we refer to Parry and Pizer (2007).

For instance, let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function and define the cost function $c : 2^N \to \mathbb{R}$ by

$$c(\emptyset) := 0 \quad \text{and} \quad c(S) := \varphi\left(\sum_{i \in S} x_i\right) \text{ for non-empty subsets } S \text{ of } N. \tag{4.1}$$

In the linear case $\varphi(x) = ax$, both allocation methods, (2.1) and (2.2), just give linear taxes $k_i = ax_i$ irrespective of the ordered partition being used. More interesting are the choices

- $\varphi(x) = \exp(ax) - 1$, where $a > 0$
- $\varphi(x) = \begin{cases} ax & \text{for } 0 \le x \le c \\ b(x-c) + ac & \text{for } x \ge c \end{cases}$, where $0 < a < b$ and $c > 0$
- $\varphi(x) = \begin{cases} a/(b-x) - a/b & \text{for } 0 \le x \le b \\ \infty & \text{for } x \ge b \end{cases}$, where $a, b > 0$.

The first two specifications of φ increase the rate for those companies that belong to a high level set σ_l. For example, for the minimal increments ordering (2.3), the rate is highest for those with the highest emissions x_i and hence, gives them stronger incentives to reduce emissions.

In the third specification, φ can take the value ∞, and our allocation schemes yield real-valued allocations only if $\sum_{i=1}^n x_i < b$. Formally, this is slightly outside of our framework. But it enforces the total emission $\sum_{i=1}^n x_i$ to stay below the level b. If, for instance, the companies are ranked according to the minimal increments ordering (2.3) and in the middle of the tax period, the total emission is on track to exceed the threshold b, those companies with the highest emissions will do everything to reduce their emissions to a level such that $\sum_{i=1}^n x_i < b$ holds at the reference date.

For general convex functions φ, one has the following result:

Proposition 4.1 *If the cost function is given by* (4.1) *for a strictly increasing convex function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* $\varphi(0) = 0$*, then the ordered contribution allocation* (2.1) *corresponding to any of the orders* (2.3)*,* (2.5) *or* (2.6) *satisfies the dummy player property.*

Proof. Let i be a player such that $c(\{i\} \cup S) - c(S) = c(i)$ for every subset S of $N \setminus \{i\}$. Then $\varphi(x_i) = \varphi(\sum_{j=1}^{n} x_j) - \varphi(\sum_{j \neq i} x_j)$. It follows that $x_i = 0$, $\sum_{j \neq i} x_j = 0$ or φ is linear on the interval $[0, \sum_{j=1}^{N} x_j]$. In all three cases one has $k_i = \varphi(x_i) = c(i)$. □

5 Computational complexity

The ordered contribution allocation (2.1) can be computed efficiently if it is easy to calculate $c(S)$ for every subset S of N. As pointed out in Section 2, for a given ordered partition $(\sigma_1, \ldots, \sigma_m)$, it requires the evaluation of the cost function only $m \leq n$ times. To obtain, for instance, the minimal increments ordering (2.3), the cost function has to be calculated at most $n(n+1)/2$ times.

The situation is different for the average ordered contribution allocation (2.2). As mentioned above, in the extreme case $\sigma_1 = N$, the average ordered contribution allocation (2.2) is the Shapley value

$$k_i = \frac{1}{n!} \sum_{\pi \in \Pi} c\{\pi(1), \ldots, \pi(i)\} - c\{\pi(1), \ldots \pi(i-1)\},$$

where Π is the set of all permutations of N and $\{\pi(1), \ldots, \pi(i-1)\}$ is understood as $\emptyset$ for $\pi(i) = 1$. This can also be written as

$$k_i = \sum_{S \subset N \setminus \{i\}} \frac{|S|!(n-1-|S|)!}{n!} (c(S \cup \{i\}) - c(S))$$

and requires the evaluation of $c(S)$ for each of the 2^n subsets S of N; see Shapley (1953). So it has complexity $\mathcal{O}(2^n)$ and is an NP-complete problem; see for instance, Deng and Papadimitriou (1994) or Faigle and Kern (1992) for more details. Several exact and approximate methods for computing the Shapley value have been studied. But most of them either presume precedence constraints on the set of players N (e.g. Faigle and Kern, 1992) or require that the cost function c has additional structure (e.g. Conitzer and Sandholm, 2004). Many of them only consider simple coalitional games such as weighted voting games; see Fatima et al. (2008) for an overview. For the risk allocation problem of Section 3 or the tax schemes proposed in Section 4 these methods are not general enough. Castro et al. (2009) have proposed a polynomial time approximation method for the Shapley value based on sampling. A similar method could be used to approximate the average ordered contribution allocation (2.2). However, since it uses sampling, it can only provide approximations that are close to the true allocation with high probability.

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