Suitability of Capital Allocations for Performance Measurement

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accepted by The Journal of Risk, July 2013

Abstract

Capital allocation principles are used in various contexts in which the risk capital or the cost of an aggregate position has to be allocated among its constituent parts. We study capital allocation principles in a performance measurement framework. We introduce the notation of suitability of allocations for performance measurement and show under different assumptions on the involved reward and risk measures that there exist suitable allocation methods. The existence of certain suitable allocation principles generally is given under rather strict assumptions on the underlying risk measure. Therefore we show, with a reformulated definition of suitability and in a slightly modified setting, that there is a known suitable allocation principle that does not require any properties of the underlying risk measure. Additionally we extend a previous characterization result from the literature from a mean-risk to a reward-risk setting. Formulations of this theory are also possible in a game theoretic setting.

Keywords Risk capital allocation, subgradient allocation, cost allocation, suitability for performance measurement, reward measure, risk measure

1 Introduction

Capital allocation principles are used in various contexts in which the risk capital or a cost of an aggregate position has to be allocated among its constituent parts. The aggregate position, for

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†We would like to thank an anonymous referee for helpful remarks and suggestions that led to an improvement of the paper.

instance, can be a portfolio of credit derivatives, the profit, cost or turnover of a firm or simply the outcome of a cooperative game. The importance of capital allocation principles in credit portfolio modeling is outlined in Kalkbrener et al. (2004). Capital allocations can even be used to construct tax schemes for pollution emission, see Cheridito and Kromer (2011). In any case one is interested in how to allocate (distribute) the aggregate cost/profit among the units which are involved in generating this cost/profit.

The main financial mathematics literature dealing with this topic follows a structural approach to capital allocations and studies the properties of specific allocation methods. Kalkbrener (2005), for instance, studies in detail the properties of the gradient allocation and places them in context with the properties of the underlying coherent risk measure. Other contributions, like Denault (2001), Fischer (2003) and McNeil et al. (2005), similarly focus on the gradient allocation and study its properties. This allocation method only exists if the underlying risk measure is Gâteaux-differentiable. This requirement is a strong one and several risk measures in general fail to be Gâteaux-differentiable. For an overview of this aspect we refer to Cheridito and Kromer (2011). For the existence of the subgradient allocation that was introduced in Delbaen (2000), we need weaker requirements. Continuity of the underlying risk measure, to name one, is a sufficient condition to ensure the existence of a subgradient allocation, see Theorem 2.4.9 in Zalinescu (2002). There are also allocation principles in the literature that do not require any properties of the risk measure and still have some desirable structural properties. Recent examples for such allocation principles are the ordered contribution allocations from Cheridito and Kromer (2011) or the with-without allocation from Merton and Perold (1993) and Matten (1996).

In this work we will focus on another aspect of capital allocation methods. These can be used as important tools in performance measurement for portfolios and firms, and consequently for portfolio optimization. The literature that studies capital allocations in this framework, to our knowledge, narrows down to the works of Tasche (2004) and Tasche (2008). He studies the gradient allocation principle based on Gâteaux-differentiable risk measures and uses mean-risk ratios as performance measures. He introduces a definition of suitability for performance measurement for capital allocations and characterizes the gradient allocation as the unique allocation that is suitable for performance measurement. This is a different approach to this topic compared to the structural approaches we previously mentioned. We will take up and extend this approach in different directions. We will show in Theorem 3.2 and Theorem 3.9, with a slightly modified definition of suitability, that we can relax the requirement of Gâteaux-differentiability of the underlying risk measure to a (local) continuity property. This indeed preserves the existence of capital allocations that are suitable for performance measurement, but we loose the uniqueness. As we will show, any subgradient allocation is suitable for performance measurement. If the underlying risk measure is Gâteaux-differentiable, the subgradient allocation is unique and reduces to the gradient allocation. In this case our result reduces to the result of Tasche (2004). In another setting, where we restrict our performance measure to portfolios that do not allow arbitrage opportunities and irrationality we are able to work with a stronger formulation of suitability and to derive in this setting an existence and uniqueness result. In the following we will work with classes of reward-risk ratios, which were recently introduced in Cheridito and Kromer (2013). These are generalizations of mean-risk ratios from Tasche (2004). We will show that a further modification of the definition of suitability for performance measurement allows us to carry over the whole approach to a cooperative game-theoretic setting and to consider cost allocation methods. This setting has the benefit that we do not require any properties of the underlying cost function to ensure existence of suitable cost allocation methods. Moreover, there is a direct connection to the risk-capital based setting that allows us to apply these results directly to risk measures without requiring the risk measure to have any specific properties.

The outline of the paper is as follows. In Section 2 we will introduce our notation and give a short overview of the capital allocation methods that will be relevant for us in the remaining work. We will work with reward-risk ratios as performance measures. These will be introduced in Section 3. In this section we will furthermore introduce the definition of suitability for performance measurement with reward-risk ratios and present our main results. In Section 4 we will modify the setting from Section 3 to a cooperative game-theoretic setting and show in this new framework the existence of capital allocations that are suitable for performance measurement.

2 Notation, definitions and important properties of risk measures and capital allocations

A financial position, the value of a firm or the outcome of a cooperative game will be modeled as a real valued random variable X from L^p, $1 \leq p \leq \infty$, on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. As usual, we identify two random variables if they agree $\mathbb{P}$-a.s. Inequalities between random variables are understood in the $\mathbb{P}$-a.s. sense.

We will work with convex and with coherent risk measures $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$. Coherent risk measures were introduced and studied in Artzner et al. (1999). We will call a functional from L^p to $\mathbb{R} \cup \{+\infty\}$ a coherent risk measure if it has the following four properties.

(M) *Monotonicity:* $\rho(X) \leq \rho(Y)$ for all $X, Y \in L^p$ such that $X \geq Y$.

(T) *Translation property:* $\rho(X+m) = \rho(X) - m$ for all $X \in L^p$ and $m \in \mathbb{R}$.

(S) *Subadditivity:* $\rho(X+Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in L^p$.

(P) *Positive homogeneity:* $\rho(\lambda X) = \lambda \rho(X)$ for all $X \in L^p$ and $\lambda \in \mathbb{R}_+$.

From the properties (S) and (P) it follows that coherent risk measures are convex, accordingly they satisfy the property

(C) *Convexity:* $\rho(\lambda X + (1-\lambda)Y) \leq \lambda \rho(X) + (1-\lambda)\rho(Y)$ for all $X, Y \in L^p$ and $\lambda \in (0,1)$.

Functionals with the properties (M), (T) and (C) are called convex risk measures. These were introduced and studied in Föllmer and Schied (2004) and Frittelli and Rosazza Gianin (2002). We will denote the domain of a convex risk measure ρ with $\operatorname{dom}\rho = \{X \in L^p \,|\, \rho(X) < \infty\}$. ρ

is a *proper* convex risk measure if the domain of ρ is nonempty. Since it only makes sense to allocate a *finite* risk capital of an aggregate position among its constituent parts we will only work with aggregate positions X from the domain of ρ. We know from the extended Namioka-Klee Theorem from Frittelli and Biagini (2010) that proper convex risk measures are continuous on the interior of their domain. From this it follows that finite valued convex risk measures on L^p are continuous on L^p, $1 \leq p \leq \infty$. In particular proper convex risk measures on L^∞ are automatically finite and continuous on L^∞. In the following we will need the concept of the subdifferential of a convex risk measure on L^p. The subdifferential of a convex risk measure at $X \in \operatorname{dom}\rho$, $1 \leq p < \infty$ is the set

$$\partial\rho(X) = \{\, \xi \in L^q \mid \rho(X+Y) \geq \rho(X) + \mathbb{E}[\xi Y] \quad \forall Y \in L^p \}, \tag{2.1}$$

where q is such that $1/q + 1/p = 1$ and L^q is the dual space of L^p. We will call ρ subdifferentiable at X if $\partial\rho(X)$ is nonempty. We know from Proposition 3.1 in Ruszczynski and Shapiro (2006) that ρ with the properties (M) and (C) is continuous and subdifferentiable on the interior of its domain. For $p = \infty$ we can consider the weak*-subdifferential introduced in Delbaen (2000)

$$\partial\rho(X) = \left\{\, \xi \in L^1 \,\middle|\, \rho(X+Y) \geq \rho(X) + \mathbb{E}[\xi Y] \quad \forall Y \in L^\infty \right\}$$

In contrast to (2.1) this set can be empty at some points in L^∞ even if the risk measure is finite valued and thus continuous on L^∞. For more details about the weak*-subdifferential we refer to Delbaen (2000). It is known for a continuous risk measure ρ that $\partial\rho(X)$ is a singleton if and only if ρ is Gâteaux-differentiable, see for instance Corollary 2.4.10 in Zalinescu (2002). Then $\mathbb{E}(\xi Y)$ for $\xi \in \partial\rho(X)$ and $Y \in L^p$ is the Gâteaux-derivative at X in the direction of Y, ie,

$$\mathbb{E}(\xi Y) = \lim_{h \to 0} \frac{\rho(X+hY) - \rho(X)}{h}. \tag{2.2}$$

All statements concerning the subdifferential of a convex risk measure obviously remain true for the superdifferential of a concave reward measure which, with respect to the properties (M)-(C) above, can be understood as the negative of a risk measure. The superdifferential of a concave reward measure $\theta : L^p \to \mathbb{R} \cup \{-\infty\}$, $1 \leq p < \infty$ at $X \in \operatorname{dom}\theta$ is the set

$$\partial\theta(X) = \{\, \psi \in L^q \mid \theta(X+Y) \leq \theta(X) + \mathbb{E}[\psi Y] \quad \forall Y \in L^p \}, \tag{2.3}$$

where q is such that $1/q + 1/p = 1$ and L^q is the dual space of L^p. As before, the weak*-superdifferential of θ at $X \in \operatorname{dom}\theta$ is considered for $p = \infty$ as the set

$$\partial\theta(X) = \left\{\, \psi \in L^1 \,\middle|\, \theta(X+Y) \leq \theta(X) + \mathbb{E}[\psi Y] \quad \forall Y \in L^\infty \right\}.$$

To be precise, we will call a map $\theta : L^p \to \mathbb{R} \cup \{-\infty\}$ a concave reward measure if it satisfies

($\overline{\mathbf{M}}$) $\theta(X) \geq \theta(Y)$ for all $X, Y \in L^p$ such that $X \geq Y$.

($\overline{\mathbf{T}}$) $\theta(X+m) = \theta(X) + m$ for all $X \in L^p$ and $m \in \mathbb{R}$.

($\overline{\mathbf{C}}$) $\theta(\lambda X + (1-\lambda)Y) \geq \lambda\theta(X) + (1-\lambda)\theta(Y)$ for all $X, Y \in L^p$ and $\lambda \in (0,1)$.

We will call it coherent reward measure if satisfies the property (P) in addition to the properties $(\overline{M}) - (\overline{C})$.

Consider now the aggregate position $X = \sum_{i=1}^{n} X_i \in \operatorname{dom}\rho,\ n \in \mathbb{N}$. It is of interest for several reasons outlined in the introduction, to allocate the aggregate risk capital $\rho(X)$ to the $X_i,\ i = 1, \ldots, n$, which are involved in generating this risk capital. We will denote a capital allocation as a vector of real numbers $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$. The simplest risk capital allocation $k \in \mathbb{R}^n$ is the individual allocation given by

$$k_i = \rho(X_i),\ i = 1, \ldots, n.$$

If the underlying risk measure is subadditive, which is property (S) of coherent risk measures, then the individual allocation overestimates the total risk, ie,

$$\rho\left(\sum_{i=1}^{n} X_i\right) \leq \sum_{i=1}^{n} \rho(X_i) = \sum_{i=1}^{n} k_i.$$

A reasonable allocation principle should satisfy the above inequality as equality. This is then usually called *full allocation* property or *efficiency* of the allocation,

$$\rho\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} k_i. \tag{2.4}$$

Since the individual allocation only depends on the risk of the individual asset or subportfolio X_i and does not take into account the risk of the whole portfolio one is certainly interested in other allocation principles. Another candidate, that additionally takes into account the risk of the aggregate position is the with-without allocation,

$$k_i = \rho(X) - \rho(X - X_i). \tag{2.5}$$

This allocation principle, known from Merton and Perold (1993) and Matten (1996), will be important in Section 4. For risk measures with property (S) both principles are related through

$$\rho(X) - \rho(X - X_i) \leq \rho(X_i),$$

such that the individual allocation is an upper bound for the with-without allocation. Nevertheless, both allocation principles in general do not satisfy the full allocation property if they are based on coherent risk measures.

If the underlying risk measure is convex and the subdifferential of ρ at X is nonempty, then for any $\xi \in \partial\rho(X)$ we can consider the subgradient allocation,

$$k_i = \mathbb{E}(\xi X_i). \tag{2.6}$$

This allocation principle was introduced in Delbaen (2000) with the weak*-subdifferential for coherent risk measures on L^∞. If $\partial\rho(X)$ is a singleton the subgradient allocation reduces to the gradient allocation which is also known as the Euler allocation, see Tasche (2008). Then $k_i = \mathbb{E}(\xi X_i)$ is the Gâteaux-derivative at X in the direction of X_i, ie,

$$k_i = \lim_{h_i \to 0} \frac{\rho(X + h_i X_i) - \rho(X)}{h_i}. \tag{2.7}$$

It is known from Euler's Theorem on positively homogeneous functions (see Tasche (2008), Theorem A.1) that it satisfies the full allocation property if the risk measure is Gâteaux-differentiable and positively homogeneous.

In the next sections we will highlight another aspect of capital allocations. We will study capital allocations with respect to their ability to give an investor the right signals in a performance measurement framework.

Before we move to the next section we will report in short the results of Tasche (2004) which form the basis for this suitability-for-performance-measurement approach and which we will generalize in the following. Let $X = \sum_{i=1}^n X_i$ be a portfolio consisting of n subportfolios X_i, $i = 1, \ldots, n$. Let $\emptyset \neq U \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be an open set and define $\rho_X : U \to \mathbb{R}$, $\rho_X(u) := \rho(\sum_{i=1}^n u_i X_i)$. With $m_i = \mathbb{E}[X_i]$ we similarly define $\mathbb{E}_X(u) = \mathbb{E}[\sum_{i=1}^n u_i X_i] = \sum_{i=1}^n u_i m_i$. Here each $u \in U$ represents a portfolio. We denote by e_i the i-th unit vector from $\mathbb{R}^n$, ie $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the i-th position of the vector. A risk capital allocation is identified in this setting by a map

$$k = (k_1, \ldots, k_n) : U \to \mathbb{R}^n.$$

Tasche (2004) considers in his approach mean-risk ratios, which are also called RORAC (return on adjusted risk capital) ratios. These are defined by $RORAC_X : U \to \mathbb{R}$,

$$RORAC_X(u) := \frac{\mathbb{E}_X(u)}{\rho_X(u)}, \tag{2.8}$$

where in Tasche (2004) irrational portfolios ($u \in U$ such that $\mathbb{E}_X(u) \leq 0$ and $\rho_X(u) > 0$) and arbitrage portfolios ($u \in U$ such that $\mathbb{E}_X(u) > 0$ and $\rho_X(u) \leq 0$) are completely excluded from the approach. In this setting with mean-risk ratios as measures of performance Tasche (2004) provides the following definition of suitability of capital allocations for performance measurement.

Definition 2.1 (see Tasche (2004), Definition 3)**.** *We say that a vector field $k : U \to \mathbb{R}^n$ is called suitable for performance measurement with ρ_X if it satisfies the following conditions:*

(i) For all $m \in \mathbb{R}^n$, $u \in U$ and $i \in \{1, \ldots, n\}$ the inequality

$$m_i \rho_X(u) > k_i(u) \mathbb{E}_X(u) \tag{2.9}$$

implies that there is an $\varepsilon > 0$ such that for all $h \in (0, \varepsilon)$ we have

$$RORAC_X(u + he_i) > RORAC_X(u) > RORAC_X(u - he_i). \tag{2.10}$$

(ii) For all $m \in \mathbb{R}^n$, $u \in U$ and $i \in \{1,\dots,n\}$ the inequality

$$m_i \rho_X(u) < k_i(u) \mathbb{E}_X(u) \tag{2.11}$$

implies that there is an $\varepsilon > 0$ such that for all $h \in (0,\varepsilon)$ we have

$$RORAC_X(u + he_i) < RORAC_X(u) < RORAC_X(u - he_i). \tag{2.12}$$

Note the strong requirement that inequalities (2.9) and (2.11) have to be true for all $m \in \mathbb{R}^n$.

With this definition of suitability for performance measurement Tasche (2004) provides the following existence and uniqueness result.

Theorem 2.2 (see Tasche (2004), Theorem 1)**.** *Let $\rho_X : U \to \mathbb{R}$ be a function that is partially differentiable in U with continuous derivatives. Let $k = (k_1,\dots,k_n) : U \to \mathbb{R}^n$ be a continuous vector field. Then the vector field k is suitable for performance measurement with ρ_X if and only if*

$$k_i(u) = \frac{\partial}{\partial u_i} \rho_X(u), \quad i = 1,\dots,n,\, u \in U. \tag{2.13}$$

If we take a closer look at the definition of suitability for performance measurement from Tasche (2004) we see that a certain type of reward capital allocation for subportfolio X_i is fixed, namely $m_i = \mathbb{E}[X_i]$ in (2.9) and (2.11), whereas the risk capital allocation $k = (k_1,\dots,k_n)$ is in the focus of investigation. This investigation then proves Theorem 2.2.

Our aim is to generalize the suitability framework of Tasche (2004) to a reward-risk setting with a more general functional $\theta : L^p \to \mathbb{R} \cup \{-\infty\}$, $1 \le p \le \infty$, instead of the expectation. But now a problem arises since Definition 2.1 can be interpreted in two ways. One way of reading Definition 2.1 is to consider

$$t_i = \theta(X_i) = \mathbb{E}[X_i] \tag{2.14}$$

to be the reward capital allocation and thus to compare the standalone performance of X_i with the performance of the portfolio $X = \sum_{i=1}^n X_i$ by considering the ratios $\theta(X_i)/k_i$ and $\theta(X)/\rho(X)$ in Definition 2.1. Since $\theta(X) = \mathbb{E}[X]$ is Gâteaux-differentiable there arises another interpretation of Definition 2.1, namely to consider the Gâteaux-derivative of $\theta(\cdot) = \mathbb{E}[\cdot]$ at X in the direction of X_i,

$$t_i = \lim_{h_i \to 0} \frac{\mathbb{E}[X + h_i X_i] - \mathbb{E}[X]}{h_i} = \mathbb{E}[X_i] \tag{2.15}$$

As we can see in (2.14) and (2.15) both concepts coincide if we consider $\theta(X) = \mathbb{E}[X]$ as a reward measure. We will study both approaches in the following section and show that we can generalize Tasche's result in both cases. The case $t_i = \theta(X_i) = \mathbb{E}[X_i]$, which corresponds to the individual reward capital allocation, will require certain structural properties of the functional θ. This will be studied at the beginning of Section 3. The Gâteaux-derivative case will be separately studied in the Subsections 3.1 and 3.2, where we will provide the direct extension of Tasche's result in Subsection 3.2. In Subsection 3.1 we won't fix any specific reward capital allocation. Instead we will focus on sufficient properties of reward-risk allocations such that the allocation is suitable for performance measurement.

In contrast to Tasche (2004) we will consider reward-risk ratios that were recently introduced in Cheridito and Kromer (2013) as measures of performance. This leads to the next section.

3 Suitability for performance measurement with RRRs

In this section we will examine the connection of reward-risk ratios to capital allocations that are suitable for performance measurement. Let us first introduce reward-risk ratios.

As performance measures in this work we consider reward-risk ratios (RRRs in short) of the form

$$\alpha(X)=\begin{cases}0, & \text{if } \theta(X)\le 0 \text{ and } \rho(X)>0,\\ +\infty, & \text{if } \theta(X)>0 \text{ and } \rho(X)\le 0,\\ \frac{\theta(X)}{\rho(X)}, & \text{else.}\end{cases} \tag{3.16}$$

This means that we will measure the performance of the aggregate position X by a ratio with a risk measure $\rho : L^p \to \mathbb{R}\cup\{+\infty\}$ in the denominator and a reward measure $\theta : L^p \to \mathbb{R}\cup\{-\infty\}$ in the numerator. Hereby $0/0$ and ∞/∞ are understood to be 0. A reward measure $\theta : L^p \to \mathbb{R}\cup\{-\infty\}$ in the setting of this section is a functional with property (P) and property

($\bar{\text{S}}$) *Superadditivity:* $\theta(X+Y)\ge\theta(X)+\theta(Y)$ for all $X,Y\in L^p$,

Classes of reward measures with these properties were recently studied in Cheridito and Kromer (2013) as robust reward measures and distorted reward measures. We will shortly repeat their definition here. Let $\mathscr{P}$ be a non-empty set of probability measures that are absolutely continuous with respect to $\mathbb{P}$. The robust reward measure is defined by

$$\theta(X)=\inf_{Q\in\mathscr{P}}\mathbb{E}_Q[X]. \tag{3.17}$$

θ defined in (3.17) obviously satisfies ($\bar{S}$) and (P). Through the set $\mathscr{P}$ we can introduce ambiguity such that the elements of the set $\mathscr{P}$ could describe the beliefs of different traders. This allows a much more general approach to the performance measurement framework than simply choosing the expectation for the numerator of the reward-risk ratio. A subclass of (3.17) are distorted reward measures. Let $\varphi : [0,1]\to[0,1]$ be a non-decreasing function satisfying $\varphi(0)=0$ and $\varphi(1)=1$. It induces the distorted probability $\mathbb{P}_\varphi[A] := \varphi\circ\mathbb{P}[A]$. If φ is convex then the Choquet integral defined by

$$\mathbb{E}_\varphi[X] := \int_0^\infty \mathbb{P}_\varphi[X>t]dt+\int_{-\infty}^0(\mathbb{P}_\varphi[X>t]-1)dt, \tag{3.18}$$

is superadditive and positively homogeneous. Thus it satisfies ($\bar{S}$) and (P). With different choices for φ we can model different attitudes of agents towards rewards. For additional information about RRRs induced by coherent or convex risk measures in conjunction with these reward measures we refer to Cheridito and Kromer (2013). Most important for us in this work are the properties ($\bar{S}$) and (P) and the existence of interesting classes of reward measures with these properties. These classes obviously contain the expectation. Thus all statements in the following sections are also true for mean-risk ratios (see e.g. (2.8)) as performance measures.

Now we will examine the connection of reward-risk-ratios to capital allocations that are suitable for performance measurement. The following definition will clarify what we mean by suitability for performance measurement with reward-risk ratios of type (3.16).

Definition 3.1. *A risk capital allocation $k \in \mathbb{R}^n$ is suitable for performance measurement at $X \in dom\,\rho$ with a reward-risk-ratio α, if for any decomposition of $X = \sum_{i=1}^n X_i$ it satisfies the following conditions:*

1. *If $\theta(X) \geq 0$ then for any $i \in \{1,\dots,n\}$ there exists $\varepsilon_i > 0$ such that for all $h_i \in (0,\varepsilon_i)$*
$$\frac{\theta(X_i)}{k_i} \geq \frac{\theta(X)}{\rho(X)} \tag{3.19}$$
implies
$$\alpha(X) \geq \alpha(X - h_i X_i) \tag{3.20}$$
and
$$\frac{\theta(X_i)}{k_i} \leq \frac{\theta(X)}{\rho(X)} \tag{3.21}$$
implies
$$\alpha(X) \geq \alpha(X + h_i X_i) \tag{3.22}$$
2. *If $\theta(X) \leq 0$ then for any $i \in \{1,\dots,n\}$ there exists $\varepsilon_i > 0$ such that for all $h_i \in (0,\varepsilon_i)$* (3.19) *implies*
$$\alpha(X + h_i X_i) \geq \alpha(X) \tag{3.23}$$
and (3.21) *implies*
$$\alpha(X - h_i X_i) \geq \alpha(X) \tag{3.24}$$

The motivation behind this definition of suitability for performance measurement is similar to the one in Tasche (2004) and Tasche (2008). Evidently, (3.19) and (3.21) relate the performance of the aggregate position X, measured by $\alpha(X)$, to the performance of each unit X_i, $i = 1,\dots,n$, which is involved in generating the risk capital $\rho(X)$. The performance of each unit X_i, $i = 1,\dots,n$, is measured by $\theta(X_i)/k_i$ (assume for the sake of simplicity that $k_i > 0$ and $\rho(X) > 0$) and only those capital allocations $k = (k_1,\dots,k_n)$ are suitable for performance measurement that send the right signals to the agent. The right signals in this context are represented by (3.20) and (3.22) (in connection to (3.19) and (3.21)) and imply not to reduce the capital of position X_i if the standalone performance of X_i is better than the performance of the aggregate position X. On the other hand a worse standalone performance of X_i should send the signal not to invest additional capital in this position. Summarized, this means

- $\alpha(X) \geq \alpha(X - h_i X_i)$ do not reduce the capital in position X_i
- $\alpha(X) \geq \alpha(X + h_i X_i)$ do not increase the capital in position X_i

- $\alpha(X+h_iX_i) \geq \alpha(X)$ increase the capital in position X_i
- $\alpha(X-h_iX_i) \geq \alpha(X)$ reduce the capital in position X_i

We will see in Subsection 3.2 that there is a stronger formulation of suitability for performance measurement, but it also requires much stronger additional properties of the underlying risk measure to guarantee the existence of suitable capital allocations.

For further statements we will need the following technical assumption on the underlying risk measure.

(A) For an aggregate position $X \in \operatorname{dom}\rho$ and any decomposition $(X_1,\dots,X_n)$ of $X=\sum_{i=1}^n X_i$ there exists $\varepsilon_i > 0$ such that for all $h_i \in (0,\varepsilon_i)$ we have

- if $\rho(X) > 0$ then $\rho(X-h_iX_i) > 0$ and $\rho(X+h_iX_i) > 0$,
- if $\rho(X) < 0$ then $\rho(X-h_iX_i) < 0$ and $\rho(X+h_iX_i) < 0$.

for any $i \in \{1,\dots,n\}$.

It should be noted that convex risk measures that are continuous at $X \in \operatorname{dom}\rho$, automatically satisfy assumption (A) at X.

Now we can examine which properties a capital allocation should have to be suitable for performance measurement with a reward-risk ratio α at $X \in \operatorname{dom}\rho$.

Theorem 3.2. *Let $\theta : L^p \to \mathbb{R}\cup\{-\infty\}$, $1 \leq p \leq \infty$, be a functional with the properties ($\bar{S}$) and (P). Let $\rho : L^p \to \mathbb{R}\cup\{+\infty\}$, $1 \leq p \leq \infty$, be a map that satisfies condition (A) in $X \in \operatorname{dom}\rho$ and let α be a reward risk ratio of type* (3.16)*. Then a risk capital allocation $k \in \mathbb{R}^n$ that satisfies the properties*

$$h_ik_i \leq \rho(X+h_iX_i) - \rho(X) \quad \textit{and} \tag{3.25}$$

$$h_ik_i \geq \rho(X) - \rho(X-h_iX_i) \tag{3.26}$$

for all $h_i \in (0,\varepsilon_i)$, ε_i from (A), $1 \leq i \leq n$, and for any decomposition $(X_1,\dots,X_n)$ of $X=\sum_{i=1}^n X_i$ is suitable for performance measurement with α at X.

Proof. Consider first the case where numerator and denominator have the same signs. Let $\theta(X) \geq 0$. With assumption (A) for ρ at $X \in \operatorname{dom}\rho$ there exists $\varepsilon_i > 0$ such that for all $h_i \in (0,\varepsilon_i)$ we have $\rho(X)\rho(X+h_iX_i) > 0$ and $\rho(X)\rho(X-h_iX_i) > 0$ for any $i \in \{1,\dots,n\}$. From ($\bar{S}$) and (P) we get $\theta(X-h_iX_i) \leq \theta(X) - h_i\theta(X_i)$. This, together with (3.19) and (3.26) leads to

$$\begin{aligned}&\theta(X-h_iX_i)\rho(X) - \theta(X)\rho(X-h_iX_i)\\ \leq&\theta(X)(\rho(X) - h_ik_i - \rho(X-h_iX_i)) \leq 0,\end{aligned}$$

which immediately implies $\alpha(X) \geq \alpha(X-h_iX_i)$. By analogous steps we get from (3.21) and (3.25) the property $\alpha(X) \geq \alpha(X+h_iX_i)$.

Now let $\theta(X) \le 0$. Then again with assumption (A) for ρ in $X \in \operatorname{dom}\rho$ there exists $\varepsilon_i > 0$ such that for all $h_i \in (0,\varepsilon_i)$ we have $\rho(X)\rho(X+h_iX_i) > 0$ and $\rho(X)\rho(X-h_iX_i) > 0$ for any $i \in \{1,\ldots,n\}$. From ($\bar{S}$), (P), (3.19) and (3.25) we get

$$\begin{aligned}&\theta(X+h_iX_i)\rho(X) - \theta(X)\rho(X+h_iX_i)\\ \ge&\theta(X)(\rho(X)+h_ik_i-\rho(X+h_iX_i)) \ge 0,\end{aligned}$$

which leads to $\alpha(X) \le \alpha(X+h_iX_i)$. By analogous steps we get from (3.21) and (3.26) the property $\alpha(X) \le \alpha(X-h_iX_i)$.

Let us now consider the case with $\theta(X) \le 0$ and $\rho(X) > 0$. Here we have $\alpha(X) = 0$ and from (3.19) it follows that

$$\alpha(X+h_iX_i) \ge \alpha(X)\frac{\rho(X)+h_ik_i}{\rho(X+h_iX_i)} = 0.$$

From (3.21) it follows that

$$\alpha(X-h_iX_i) \ge \alpha(X)\frac{\rho(X)-h_ik_i}{\rho(X-h_iX_i)} = 0.$$

The remaining relevant case, in which $\theta(X) > 0$ and $\rho(X) < 0$, leads to $\alpha(X) = +\infty$ and since $\alpha(X+h_iX_i) \le +\infty$ and $\alpha(X-h_iX_i) \le +\infty$ we have finished the proof. □

In Theorem 3.2, except for condition (A), we didn't require any further properties like coherency or convexity from the risk measure ρ. We have only used the superadditivity and the positive homogeneity of θ, which are the properties ($\bar{S}$) and (P). The result nevertheless does not guarantee the existence of allocations that satisfy the required properties of this theorem. However it gives us a suitability-criterion, which is easy to verify and thus gives us the opportunity to check whether a capital allocation is suitable for performance measurement or not, without requiring the risk measure to have any properties except (A). For instance we can immediately see that the with-without allocation does not satisfy both of the properties (3.25) and (3.26). For the following corollary that tells us which capital allocations satisfy (3.25) and (3.26) and thus are suitable for performance measurement we will need the nonemptiness of the subdifferential of ρ at X.

Corollary 3.3. *Let $\theta : L^p \to \mathbb{R}\cup\{-\infty\}$, $1 \le p \le \infty$, be a be a functional with the properties ($\bar{S}$) and (P). Let $\rho : L^p \to \mathbb{R}\cup\{+\infty\}$, $1 \le p \le \infty$, be a convex risk measure with nonempty subdifferential at $X \in \operatorname{dom}\rho$. Let α be a reward-risk ratio of type* (3.16)*. If (A) is true for ρ at $X \in \operatorname{dom}\rho$ then for any $\xi \in \partial\rho(X)$ the subgradient risk capital allocation $k \in \mathbb{R}^n$ defined by*

$$k_i := \mathbb{E}[\xi X_i], \quad \textit{for any} \quad i \in \{1,\ldots,n\} \tag{3.27}$$

is suitable for performance measurement with α at X.

Proof. Since $\partial\rho(X)$ is nonempty there exits $\xi \in \partial\rho(X)$ for $X \in \operatorname{dom}\rho \subseteq L^p$, $1 \leq p \leq \infty$ such that $k_i = \mathbb{E}[\xi X_i]$, $i = 1, \ldots, n$, obviously satisfies (3.25) for any decomposition of X. Furthermore for any decomposition $(X_1, \ldots, X_n)$ of $X = \sum_{i=1}^n X_i$ it satisfies

$$\rho(X - h_i X_i) - \rho(X) \geq \mathbb{E}[\xi(-h_i X_i)]$$

for any $h_i \in (0, \varepsilon_i)$, $i = 1, \ldots, n$. This is equivalent to (3.26). Thus with Theorem 3.2 for any $\xi \in \partial\rho(X)$ any subgradient capital allocation $k_i = \mathbb{E}[\xi X_i]$, $\xi \in \partial\rho(X)$, $i = 1, \ldots, n$, is suitable for performance measurement with α at X. □

As outlined in Section 2 we know that convex risk measures on L^p, $1 \leq p < \infty$, are continuous and subdifferentiable on the interior of their domain. Thus if the aggregate position X is from the interior of the domain of ρ condition (A) and the nonemptiness of the subdifferential of ρ at X are automatically satisfied. This leads to the next corollary.

Corollary 3.4. *Let $\theta : L^p \to \mathbb{R} \cup \{-\infty\}$, $1 \leq p < \infty$ be a functional with the properties ($\bar{S}$) and (P). Let $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$, $1 \leq p < \infty$ be a convex risk measure and let α be a reward-risk ratio of type* (3.16). *If the aggregate position $X = \sum_{i=1}^n X_i$ is from the interior of the domain of ρ then the subdifferential of ρ at X is nonempty and any subgradient risk capital allocation defined in* (3.27) *is suitable for performance measurement with α at X.*

If we directly assume that the underlying convex risk measure is continuous at the aggregate position X, then assumption (A) is satisfied, the subdifferential of ρ at X is nonempty and thus the subgradient allocation exists and is suitable for performance measurement with a reward-risk ratio α. Since the continuity of a convex risk measure ρ on L^p follows from the finiteness of ρ on L^p, $1 \leq p \leq \infty$ we can formulate the following corollary.

Corollary 3.5. *Let $\theta : L^p \to \mathbb{R}$, $1 \leq p < \infty$ be a functional with the properties ($\bar{S}$) and (P). Let $\rho : L^p \to \mathbb{R}$, $1 \leq p < \infty$ be a convex risk measure and let α be a reward-risk ratio of type* (3.16). *Then the subdifferential of ρ at X is nonempty for any $X \in L^p$ and any subgradient risk capital allocation is suitable for performance measurement with α at any $X \in L^p$.*

As already mentioned in Section 2, Gâteaux-differentiability of a continuous ρ at X leads to the uniqueness of the subgradient of ρ at X. In this case the Gâteaux-derivative is the only element of the subdifferential of ρ at X and thus the capital allocation defined in (2.7) is suitable for performance measurement with a reward-risk ratio α at X. This allows us to formulate the final corollary from Theorem 3.2.

Corollary 3.6. *Let $\theta : L^p \to \mathbb{R} \cup \{-\infty\}$, $1 \leq p \leq \infty$ be a functional with the properties ($\bar{S}$) and (P). Let α be a reward-risk ratio of type* (3.16). *If the risk measure $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$, $1 \leq p \leq \infty$ is continuous and Gâteaux-differentiable at the aggregate position X, then the gradient risk capital allocation, defined by*

$$k_i = \lim_{h_i \to 0} \frac{\rho(X + h_i X_i) - \rho(X)}{h_i}, \quad i = 1, \ldots, n, \tag{3.28}$$

exists and is suitable for performance measurement with α at X.

Proof. This follows directly from Corollary 3.3 and the reasoning from Section 2. □

Example 3.7. *Assume that the underlying probability space is non-atomic. Distortion-exponential risk measures from Tsanakas and Desli (2003) are defined as*

$$\rho_{\psi,a}(X) = \frac{1}{a} \ln \mathbb{E}_{\psi}(\exp(-aX)), \tag{3.29}$$

where $a > 0$ and $\mathbb{E}_{\psi}[\cdot]$ is the Choquet integral from (3.18) *with a concave map ψ. Let ψ be differentiable. Then we know from Proposition 2 in Tsanakas (2009) that $\rho_{\psi,a}$ is Gâteaux-differentiable at X, if and only if F_X^{-1} is strictly increasing. The gradient risk capital allocation in this case is given by*

$$k_i = \frac{\mathbb{E}[-X_i e^{-aX} \psi'(F_X(X))]}{\mathbb{E}[e^{-aX} \psi'(F_X(X))]},$$

where F_X^{-1} is the quantile function

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}.$$

Note that with $\psi(t) = t$ this class of risk measures covers the entropic risk measure.

If we consider distortion risk measures that are defined as Choquet-integrals with a concave map ψ, see (3.18)*, then from Carlier and Dana (2003) the following result holds. Let ψ be differentiable. Then $\rho_{\psi}(X) = \mathbb{E}_{\psi}[-X]$ is Gâteaux-differentiable if and only if F_X^{-1} is strictly increasing. In this case the gradient risk capital allocation is given by*

$$k_i = \mathbb{E}[-X_i \psi'(F_X(X))].$$

As we have mentioned in Section 2 the interpretation $t_i = \theta(X_i) = \mathbb{E}[X_i]$ of the reward capital allocation in Definition 2.1 is not the only one possible. Thus we will study the other case

$$t_i = \lim_{h_i \to 0} \frac{\mathbb{E}[X + h_i X_i] - \mathbb{E}[X]}{h_i} = \mathbb{E}[X_i]$$

in the following two subsections. In Subsection 3.1 we won't fix a specific reward capital allocation in the definition of suitability for performance measurement. In case of Gâteaux-differentiable reward and risk measures this will nevertheless lead to gradient reward and risk capital allocations. In Subsection 3.2 we will follow a similar approach as Tasche (2004) (see Definition 2.1), fix the gradient reward capital allocation in the numerator and show that the only risk capital allocation that is suitable for performance measurement is the gradient risk capital allocation.

3.1 Suitability of reward-risk allocations for performance measurement

One way of dealing with portfolios that lead to *arbitrage opportunities* ($\theta(X) > 0$ and $\rho(X) \leq 0$) or are *irrational* ($\theta(X) \leq 0$, $\rho(X) > 0$) is to set the performance measure for such portfolios to $+\infty$ respectively 0. This is the way we have followed in the previous section. Another way is to restrict the performance measure to portfolios that do not allow for such possibilities. This is the direction we will follow in this subsection. Let $\mathscr{X}$ be a subset of L^p, $1 \leq p \leq \infty$, such that for each $X \in \mathscr{X}$ either

$$\theta(X) > 0 \quad \text{and} \quad \rho(X) > 0 \tag{3.30}$$

or

$$\theta(X) < 0 \quad \text{and} \quad \rho(X) < 0 \tag{3.31}$$

Here we do not fix any specific reward capital allocation in the definition of suitability for performance measurement as we have done in Definition 3.1. Instead we will modify the definition of suitability to incorporate an arbitrary reward capital allocation $t = (t_1, \dots, t_n)$ and show in the following results the existence of reward-risk capital allocations (t, k) that are suitable for performance measurement.

Definition 3.8. *A reward-risk allocation $(t,k) \in \mathbb{R}^n \times \mathbb{R}^n$ is suitable for performance measurement at $X \in \mathscr{X} \cap \operatorname{dom}\rho \cap \operatorname{dom}\theta$ with a reward-risk ratio α if for any decomposition of $X = \sum_{i=1}^n X_i$ it satisfies the following conditions:*

1. *If $\theta(X) > 0$ then for any $i \in \{1, \dots, n\}$ there exists $\varepsilon_i > 0$ such that for all $h_i \in (0, \varepsilon_i)$*

$$\frac{t_i}{k_i} > \frac{\theta(X)}{\rho(X)} \tag{3.32}$$

implies

$$\alpha(X) > \alpha(X - h_i X_i) \tag{3.33}$$

and

$$\frac{t_i}{k_i} < \frac{\theta(X)}{\rho(X)} \tag{3.34}$$

implies

$$\alpha(X) > \alpha(X + h_i X_i). \tag{3.35}$$

2. *If $\theta(X) < 0$ then for any $i \in \{1, \dots, n\}$ there exists $\varepsilon_i > 0$ such that for all $h_i \in (0, \varepsilon_i)$* (3.32) *implies*

$$\alpha(X + h_i X_i) > \alpha(X) \tag{3.36}$$

and (3.34) *implies*

$$\alpha(X - h_i X_i) > \alpha(X). \tag{3.37}$$

The interpretation of this definition of suitability for performance measurement remains the same as for Definition 3.1. We have only changed the measure of the standalone performance of the units X_i, $i = 1, \dots, n$. In Definition 3.1 we measure the standalone performance of X_i by $\theta(X_i)/k_i$ whereas in Definition 3.8 we change this to the ratio of reward-risk allocations t_i/k_i, $i = 1, \dots, n$. Again it is assumption (A) that allows us to formulate and prove a modification of Theorem 3.2 to the suitability setting we have introduced in Definition 3.8.

Theorem 3.9. *Let $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$, $1 \le p \le \infty$, be a map that satisfies condition (A) for $X \in \mathscr{X} \cap \text{dom}\,\rho \cap \text{dom}\,\theta$. Then any reward-risk allocation $(t,k) \in \mathbb{R}^n \times \mathbb{R}^n$ that satisfies the properties*

$$h_i t_i \ge \theta(X + h_i X_i) - \theta(X), \tag{3.38}$$
$$h_i t_i \le \theta(X) - \theta(X - h_i X_i), \tag{3.39}$$
$$h_i k_i \le \rho(X + h_i X_i) - \rho(X), \tag{3.40}$$
$$h_i k_i \ge \rho(X) - \rho(X - h_i X_i), \tag{3.41}$$

for all $h \in (0, \varepsilon_i)$, ε_i from (A), $i = 1, \ldots, n$, and for any decomposition $(X_1, \ldots, X_n)$ of $X = \sum_{i=1}^n X_i$ is suitable for performance measurement with α at X according to Definition 3.8.

Proof. The proof is similar to the proof of Theorem 3.2. However, here we don't rely on the properties $(\overline{S})$ and (P) of θ. Instead we can use the properties (3.38) and (3.39) that are required from the reward allocation. With assumption (A) for ρ at $X \in \mathscr{X} \cap \text{dom}\,\rho \cap \text{dom}\,\theta$ there exists $\varepsilon_i > 0$ such that for all $h_i \in (0, \varepsilon_i)$ we have $\rho(X)\rho(X + h_i X_i) > 0$ and $\rho(X)\rho(X - h_i X_i) > 0$ for any $i = 1, \ldots, n$. Let $\theta(X) > 0$, then from (3.39) we get

$$\begin{aligned} &\theta(X - h_i X_i)\rho(X) - \theta(X)\rho(X - h_i X_i) \\ \le &\theta(X)(\rho(X) - \rho(X - h_i X_i)) - h_i t_i \rho(X). \end{aligned}$$

Now (3.41) gives us

$$\theta(X)(\rho(X) - \rho(X - h_i X_i)) - h_i t_i \rho(X) \le h_i(\theta(X) k_i - \rho(X) t_i).$$

With (3.32) this leads to $\alpha(X) > \alpha(X - h_i X_i)$. Similarly we get from (3.38) and (3.40) the inequality

$$\theta(X + h_i X_i)\rho(X) - \theta(X)\rho(X + h_i X_i) \le h_i(t_i \rho(X) - k_i \theta(X)).$$

With (3.34) this leads to $\alpha(X) > \alpha(X + h_i X_i)$.

Now let $\theta(X) < 0$. Since $X \in \mathscr{X}$ this means that $\rho(X) < 0$. From this, (3.38) and (3.40) we get

$$\begin{aligned} &\theta(X + h_i X_i)\rho(X) - \theta(X)\rho(X + h_i X_i) \\ \ge &h_i t_i \rho(X) + \theta(X)(\rho(X) - \rho(X + h_i X_i)) \\ \ge &h_i(t_i \rho(X) - k_i \theta(X)). \end{aligned}$$

Together with (3.32) this leads to $\alpha(X + h_i X_i) > \alpha(X)$. By analogous steps we get from (3.39) and (3.41)

$$\theta(X - h_i X_i)\rho(X) - \theta(X)\rho(X - h_i X_i) \ge h_i(k_i \theta(X) - t_i \rho(X))$$

which leads with (3.34) to $\alpha(X - h_i X_i) > \alpha(X)$. □

As we have seen in Theorem 3.9 it was possible to carry over the suitability for performance measurement approach from the beginning of Section 3 to a reward-risk capital allocation setting. In a similar way as in Corollary 3.3 we can provide conditions for θ and ρ that guarantee the existence of suitable reward-risk capital allocations. The next corollary summarizes the analogue statements to the results that we have presented in Corollary 3.3 to Corollary 3.6. These can be immediately recovered from the previous results.

Corollary 3.10. *Let $\theta : \mathscr{X} \to \mathbb{R} \cup \{-\infty\}$ be a concave reward measure and let $\rho : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ be a convex risk measure. Then each of the following conditions is sufficient for the reward-risk allocation $(t,k) \in \mathbb{R}^n \times \mathbb{R}^n$ to exist and to be suitable for performance measurement with the reward-risk ratio α at X according to Definition 3.8.*

(a) *ρ satisfies (A) at $X \in \mathscr{X} \cap dom\, \rho \cap dom\, \theta$, the sets $\partial\theta(X)$ and $\partial\rho(X)$ are nonempty and for any $\xi \in \partial\rho(X)$ and any $\psi \in \partial\theta(X)$ the reward-risk capital allocation $(t,k) \in \mathbb{R}^n \times \mathbb{R}^n$ is given by*

$$t_i := \mathbb{E}[\psi X_i] \quad \textit{and} \quad k_i := \mathbb{E}[\xi X_i] \tag{3.42}$$

for any $i \in \{1,\dots,n\}$.

(b) *$X \in \mathscr{X} \cap int(dom\, \rho) \cap int(dom\, \theta) \subset L^p$, $1 \leq p < \infty$ and the reward-risk capital allocation (t,k) is of type* (3.42) *with $\xi \in \partial\rho(X)$ and $\psi \in \partial\theta(X)$.*

(c) *θ and ρ are finite valued for any $X \in L^p$, $1 \leq p < \infty$ and the reward-risk capital allocation (t,k) is of type* (3.42) *with $\xi \in \partial\rho(X)$ and $\psi \in \partial\theta(X)$.*

(d) *θ and ρ are continuous and Gâteaux-differentiable at the aggregate position $X \in \mathscr{X}$ and the reward-risk capital allocation (t,k) is given by*

$$k_i = \lim_{h_i \to 0} \frac{\rho(X + h_i X_i) - \rho(X)}{h_i},$$
$$t_i = \lim_{h_i \to 0} \frac{\theta(X + h_i X_i) - \theta(X)}{h_i}$$

for any $i \in \{1,\dots,n\}$.

Condition (d) from Corollary 3.10 allows us to continue Example 3.7.

Example 3.11. *In the previous results we have included reward capital allocations to our approach. Thus we can continue Example 3.7, where we have considered distortion risk measures, to incorporate distortion reward measures as in* (3.18)*. Let the underlying probability space be atomless. Let φ be a convex and ψ be a concave distortion function. Let both functions be differentiable. Then $\theta_\varphi(X) = \mathbb{E}_\varphi[X]$ is a Gâteaux-differentiable coherent reward measure and $\rho_\psi(X) = \mathbb{E}_\psi[-X]$ is a Gâteaux-differentiable coherent risk measure. With this, according to Definition 3.8, we have to check whether we have*

$$\frac{t_i}{k_i} = \frac{\mathbb{E}[X_i \varphi'(1 - F_X(X))]}{\mathbb{E}[-X_i \psi'(F_X(X))]} > \frac{\mathbb{E}_\varphi[X]}{\mathbb{E}_\psi[-X]} \quad \textit{or} \quad \frac{t_i}{k_i} = \frac{\mathbb{E}[X_i \varphi'(1 - F_X(X))]}{\mathbb{E}[-X_i \psi'(F_X(X))]} < \frac{\mathbb{E}_\varphi[X]}{\mathbb{E}_\psi[-X]}$$

to make a decision about the investment in subportfolio X_i.

Gâteaux-differentiable risk and reward measures allow us to formulate stronger results regarding suitability for performance measurement. Thus we will deal with this aspect in a separate subsection.

3.2 Suitability for performance measurement with Gâteaux-differentiable reward and risk measures

As in the previous subsection we will restrict our attention to portfolios that do not allow for arbitrage opportunities or are irrational. Thus let $\emptyset \neq U \subset \mathbb{R}^n$ be an open set such that for each $u \in U$ either

$$\rho_X(u) := \rho\left(\sum_{i=1}^{n} u_i X_i\right) > 0 \quad \text{and} \quad \theta_X(u) := \theta\left(\sum_{i=1}^{n} u_i X_i\right) > 0, \tag{3.43}$$

or

$$\rho_X(u) := \rho\left(\sum_{i=1}^{n} u_i X_i\right) < 0 \quad \text{and} \quad \theta_X(u) := \theta\left(\sum_{i=1}^{n} u_i X_i\right) < 0. \tag{3.44}$$

Furthermore let $U \subset \mathbb{R}^n$ be such that $\rho\left(\sum_{i=1}^{n} u_i X_i\right) \in \mathbb{R}$ and $\theta\left(\sum_{i=1}^{n} u_i X_i\right) \in \mathbb{R}$. Analogously to the previous section we define

$$\alpha_X(u) := \frac{\theta_X(u)}{\rho_X(u)} \tag{3.45}$$

and denote reward and risk capital allocations, which are vector fields here, by

$$t = (t_1, \dots, t_n) : U \to \mathbb{R}^n$$
$$k = (k_1, \dots, k_n) : U \to \mathbb{R}^n.$$

In this setting, as can be seen from the notation above, each $u \in U$ represents a portfolio. Here we fix an arbitrary decomposition of X and the only assumption that will be needed for all the following results is that θ_X and ρ_X are partially differentiable with continuous derivatives in some $u \in U$. This corresponds to Gâteaux-differentiability of θ and ρ at some aggregate position $\sum_{i=1}^{n} u_i X_i$.

Now, following the lines of Tasche (2004), we can state a stronger definition of suitability for performance measurement that fits this framework.

Definition 3.12. *We deem a reward-risk allocation principle* (t,k) suitable for performance measurement with α_X *at* $u \in U$ *if the following two conditions hold:*

1. *For any* $i \in \{1, \dots, n\}$ *there is some* $\varepsilon_i > 0$ *such that*

$$\frac{t_i(u)}{k_i(u)} > \frac{\theta_X(u)}{\rho_X(u)} \tag{3.46}$$

implies

$$\alpha_X(u + se_i) > \alpha_X(u) > \alpha_X(u - se_i). \tag{3.47}$$

for all $s \in (0, \varepsilon_i)$.

2. *For any $i \in \{1,\dots,n\}$ there is some $\varepsilon_i > 0$ such that*

$$\frac{t_i(u)}{k_i(u)} < \frac{\theta_X(u)}{\rho_X(u)} \tag{3.48}$$

implies

$$\alpha_X(u - se_i) > \alpha_X(u) > \alpha_X(u + se_i). \tag{3.49}$$

for all $s \in (0, \varepsilon_i)$.

Note the difference between our definition of suitability for performance measurement with α_X and the Definition 3 of Tasche (2004). Definition 3 of Tasche (2004) requires the capital allocation k to satisfy (3.46) and (3.48) for all linear functions θ with $\theta(0) = 0$ and all portfolios $u \in U$. In our formulation we restrict this requirement to a specific reward-risk ratio α_X and a specific portfolio $u \in U$.

Furthermore note the difference between Definition 3.12 and the Definitions 3.1 and 3.8. Definitions 3.1 and 3.8 have a one-sided nature in (3.20) and (3.22), respectively (3.33) and (3.35), whereas in Definition 3.12 we are able to formulate (3.47) and (3.49) as

$$\alpha_X(u + se_i) > \alpha_X(u) > \alpha_X(u - se_i)$$
$$\alpha_X(u - se_i) > \alpha_X(u) > \alpha_X(u + se_i)$$

This is justified by the one-sided nature of the sub-, respectively superdifferential, of θ and ρ that was needed to show existence of suitable capital allocations without the requirement of Gâteaux-differentiability of θ and ρ. On the other hand, if we assume that θ and ρ are Gâteaux-differentiable then the two-sided nature of the Gâteaux-derivative allows us to formulate a stronger version of suitability for performance measurement. This is the formulation of Definition 3.12.

Now we can state and prove in a different way our version of Theorem 1 in Tasche (2004). This version fits the requirements of our definition of suitability for performance measurement with α_X in $u \in U$ in conjunction with reward-risk ratios instead of mean-risk ratios. The following result characterizes the gradient risk capital allocation in this performance measurement framework, if we fix the reward capital allocation to be the gradient reward capital allocation.

Theorem 3.13. *Let $\emptyset \neq U \subset \mathbb{R}^n$ be an open set such that* (3.43) *and* (3.44) *are true and let $\theta_X : U \to \mathbb{R}$ and $\rho_X : U \to \mathbb{R}$ be partially differentiable in $u \in U$ with continuous derivatives. Furthermore let $t = (t_1,\dots,t_n) : U \to \mathbb{R}^n$ and $k = (k_1,\dots,k_n) : U \to \mathbb{R}^n$ be continuous in $u \in U$. Let α_X be a reward-risk ratio of type* (3.45)*. Then we have the following two statements:*

(a) The gradient reward-risk allocation (t,k), defined by

$$t_i(u) := \frac{\partial \theta_X(u)}{\partial u_i}, \tag{3.50}$$

$$k_i(u) := \frac{\partial \rho_X(u)}{\partial u_i}, \quad i = 1,\dots,n,\ u \in U \tag{3.51}$$

is suitable for performance measurement with α_X at $u \in U$.

(b) *Let the reward capital allocation* $t = (t_1, \ldots, t_n) : U \to \mathbb{R}^n$ *be given by the gradient reward capital allocation from* (3.50). *If a risk capital allocation* $k : U \to \mathbb{R}^n$ *is suitable for performance measurement with* α_X *in* $u \in U$ *for any* θ_X *that is partially differentiable in* $u \in U$*, then it is the gradient risk capital allocation from* (3.51).

Proof. Let us start with (a). Consider the partial derivative of α_X in u_i,

$$\frac{\partial \alpha_X(u)}{\partial u_i} = \rho_X(u)^{-2} \left(\frac{\partial \theta_X(u)}{\partial u_i} \rho_X(u) - \theta_X(u) \frac{\partial \rho_X(u)}{\partial u_i} \right) \tag{3.52}$$

With (3.50) and (3.51), we get

$$\frac{\partial \alpha_X(u)}{\partial u_i} = \rho_X(u)^{-2} \left(t_i(u) \rho_X(u) - \theta_X(u) k_i(u) \right).$$

(3.46) now leads to $\left(\frac{\partial \theta_X(u)}{\partial u_i} \rho_X(u) - \theta_X(u) k_i(u) \right) > 0$ and therefore it follows that

$$\frac{\partial \alpha_X(u)}{\partial u_i} > 0.$$

This gives us the existence of $\varepsilon_i > 0$ such that (3.47) holds for all $s \in (0, \varepsilon_i)$, $i = 1, \ldots, n$. The proof of condition 2 in Definition 3.12 works analogously and the reward-risk capital allocation from (3.50) and (3.51) is suitable for performance measurement with α_X in $u \in U$.

Let us now prove (b). Fix the reward capital allocation to be the gradient reward capital allocation. Let k be suitable for performance measurement for any θ_X that is partially differentiable in $u \in U$. We want to show, that k corresponds to (3.51). Assume without any restriction, that

$$k_i(u) < \frac{\partial \rho_X(u)}{\partial u_i}, \quad i \in \{1, \ldots, n\}. \tag{3.53}$$

Since k is suitable for performance measurement with α_X in $u \in U$ for **any** $\theta_X : U \to \mathbb{R}$ that is partially differentiable in $u \in U$ we can simply choose $\theta_X^t : U \to \mathbb{R}$ to be $\theta_X^t(u) = t\rho_X(u)$ for $t > 0$. This function is by assumption partially differentiable in $u \in U$ for each $t > 0$. The choice of θ provides us now with the following

$$\begin{aligned} \frac{\partial \theta_X^t(u)}{\partial u_i} \rho_X(u) - k_i(u) \theta_X^t(u) &= t \frac{\partial \rho_X(u)}{\partial u_i} \rho_X(u) - k_i(u) t \rho_X(u) \\ &= t \rho_X(u) \left(\frac{\partial \rho_X(u)}{\partial u_i} - k_i(u) \right). \end{aligned}$$

Now, dependent on the sign of ρ_X, from (3.53) we either have

$$t \rho_X(u) \left(\frac{\partial \rho_X(u)}{\partial u_i} - k_i(u) \right) > 0$$

which should lead to

$$\alpha_X(u + se_i) > \alpha_X(u) > \alpha_X(u - se_i),$$

or we have

$$t\rho_X(u)\left(\frac{\partial \rho_X(u)}{\partial u_i} - k_i(u)\right) < 0,$$

which should lead to

$$\alpha_X(u+se_i) < \alpha_X(u) < \alpha_X(u-se_i),$$

since k is suitable for performance measurement with α_X in $u \in U$. But we have $\alpha_X(u) = \frac{\theta_X^t(u)}{\rho_X(u)} = \frac{t\rho_X(u)}{\rho_X(u)} = t$ and hence

$$\alpha_X(u+se_i) = \alpha_X(u) = \alpha_X(u-se_i) \quad \text{or} \quad \frac{\partial \alpha_X(u)}{\partial u_i} = 0,$$

which is a contradiction to the suitability assumption. □

4 Suitability for performance measurement with non-divisible portfolios and connections to a game theoretic context

Let $N = \{1,\dots,n\}$ and denote by 2^N the set of all possible subsets of N. In this section we will work with the functions

$$\vartheta(S) := \theta\left(\sum_{i\in S} X_i\right) \quad \text{and} \quad c(S) := \rho\left(\sum_{i\in S} X_i\right) \tag{4.54}$$

where S is a subset of N and the maps $\theta : L^p \to \mathbb{R}$ and $\rho : L^p \to \mathbb{R}$, $1 \le p \le \infty$, represent, with the notation from the previous section, the reward and the risk measure. We have chosen this approach to incorporate a study of suitability for performance measurement in a context where it is not possible to further divide the subportfolios X_i of the portfolio $X = \sum_{i=1}^n X_i$. Here it is only possible to add or to remove the whole subportfolio X_i to or from the portfolio X.

Following the approach in Subsection 3.1 and 3.2 we will focus only on on portfolios that do not allow for arbitrage gains or are irrational. Thus let $G \subset 2^N$ be such that for each $S \in G$ we either have

$$\theta\left(\sum_{i\in S} X_i\right) > 0 \quad \text{and} \quad \rho\left(\sum_{i\in S} X_i\right) > 0$$

or

$$\theta\left(\sum_{i\in S} X_i\right) < 0 \quad \text{and} \quad \rho\left(\sum_{i\in S} X_i\right) < 0.$$

Thus the functions ϑ and c are maps from G to $\mathbb{R}$ and we will work with the convention $\vartheta(\emptyset) = c(\emptyset) = 0$. In this context the reward-risk ratio becomes a map $\gamma : G \to (0,\infty)$ which is given by

$$\gamma(S) = \frac{\vartheta(S)}{c(S)} = \frac{\theta\left(\sum_{i\in S} X_i\right)}{\rho\left(\sum_{i\in S} X_i\right)}$$

and the risk capital allocation is a function $\kappa : G \to \mathbb{R}^n$ where $\kappa_i(S)$ specifies how much risk capital is allocated to each subportfolio X_i of the portfolio $\sum_{i\in S} X_i$. With this notation we can state the appropriate definition of suitability for performance measurement with γ in this context.

Definition 4.1. *We will call a capital allocation κ suitable for performance measurement with γ and a portfolio $\sum_{i\in S} X_i$ of non-divisible subportfolios if the following two conditions hold:*

1. *For any $i \in N \setminus S$*
$$\frac{\theta(X_i)}{\kappa_i(S)} = \frac{\vartheta(\{i\})}{\kappa_i(S)} > \frac{\vartheta(S)}{c(S)} = \frac{\theta\left(\sum_{j\in S} X_j\right)}{\rho\left(\sum_{j\in S} X_j\right)} \tag{4.55}$$
implies that
$$\frac{\theta\left(\sum_{j\in S\cup\{i\}} X_j\right)}{\rho\left(\sum_{j\in S\cup\{i\}} X_j\right)} = \gamma(S\cup\{i\}) > \gamma(S) = \frac{\theta\left(\sum_{j\in S} X_j\right)}{\rho\left(\sum_{j\in S} X_j\right)}. \tag{4.56}$$
2. *For any $i \in N \setminus S$*
$$\frac{\theta(X_i)}{\kappa_i(S)} = \frac{\vartheta(\{i\})}{\kappa_i(S)} < \frac{\vartheta(S)}{c(S)} = \frac{\theta\left(\sum_{j\in S} X_j\right)}{\rho\left(\sum_{j\in S} X_j\right)} \tag{4.57}$$
implies that
$$\frac{\theta\left(\sum_{j\in S\cup\{i\}} X_j\right)}{\rho\left(\sum_{j\in S\cup\{i\}} X_j\right)} = \gamma(S\cup\{i\}) < \gamma(S) = \frac{\theta\left(\sum_{j\in S} X_j\right)}{\rho\left(\sum_{j\in S} X_j\right)}. \tag{4.58}$$

This definition of suitability differs from our previous definitions. Here, starting from a portfolio $\sum_{i\in S} X_i$, we are interested in risk allocations that give us the right information whether to add a specific subportfolio X_i, $i \in N \setminus S$, to our portfolio $\sum_{i\in S} X_i$ or not. If the standalone performance of subportfolio X_i (measured by $\vartheta(\{i\})/\kappa_i(S)$) is better than the performance of the whole portfolio $\sum_{i\in S} X_i$ then the subportfolio X_i should be added to the portfolio to improve the performance of the portfolio. On the other hand, if the standalone performance of the subportfolio X_i, $i \in N \setminus S$, is worse than the performance of $\sum_{i\in S} X_i$ then adding this subportfolio to the portfolio should worsen the performance of $\sum_{i\in S} X_i$.

The connection of this approach to a game theoretic setting can be established by considering the set N to be a finite set of players such that the subsets $S \subset N$ can be interpreted as coalitions of players. Then $c(S)$ represents the cost and $\vartheta(S)$ represents the gain the coalition of players S generates in a game. From the viewpoint of the coalition of players S we are interested in cost allocation principles that give us the right information whether to add player i to our coalition S or not. In this context γ provides a possible tool and Definition 4.1 provides the criteria that are needed to make such a decision.

For details about cooperative games we refer the reader to Shapley (1953), Hart and Kurz (1983), Faigle and Kern (1992) and Branzei et al. (2005).

In cooperative game theory reward functions like ϑ (characteristic functions of a game) are often assumed to satisfy certain properties like superadditivity, which is in this framework the property
$$\vartheta(S\cup T) \geq \vartheta(S) + \vartheta(T) \quad \text{for all} \quad S, T \subseteq N,\ S\cap T = \emptyset.$$

For the following proposition we will assume that the reward function ϑ satisfies a stronger property, namely the additivity property
$$\vartheta(S\cup\{i\}) = \vartheta(S) + \vartheta(\{i\}) \tag{4.59}$$

for all $i \in N$ and all $S \subseteq N \setminus \{i\}$. We can see immediately that this property is equivalent to

$$\vartheta(S \cup T) = \vartheta(S) + \vartheta(T) \quad \text{for any } S, T \subseteq N, S \cap T = \emptyset.$$

This property enables us to state conditions for allocations to be suitable for performance measurement with γ and a coalition of players $S \in G$ in a cooperative game-theoretic framework. But we will not need ϑ to satisfy property (4.59) for all $i \in N$ and all $S \subseteq N \setminus \{i\}$. We will only need the restriction to a specific coalition of players $S \in G$ and any $i \in N \setminus S$.

Proposition 4.2. *If the reward function ϑ satisfies the additivity property* (4.59) *for the coalition of players $S \in G$ and any $i \in N \setminus S$ then any cost allocation κ that satisfies*

$$\kappa_i(S) \geq c(S \cup \{i\}) - c(S) \quad \text{if } c(S \cup \{i\}) > 0 \quad \text{and} \tag{4.60}$$

$$\kappa_i(S) \leq c(S \cup \{i\}) - c(S) \quad \text{if } c(S \cup \{i\}) < 0 \tag{4.61}$$

for any $i \in N \setminus S$ is suitable for performance measurement with γ and the coalition of players $S \in G$.

Proof. From (4.55) together with the additivity property (4.59) of ϑ we get

$$\begin{aligned}
\gamma(S \cup \{i\}) - \gamma(S) &= \frac{\vartheta(S \cup \{i\})c(S) - \vartheta(S)c(S \cup \{i\})}{c(S \cup \{i\})c(S)} \\
&> \frac{\vartheta(S)[c(S) + \kappa_i(S) - c(S \cup \{i\})]}{c(S \cup \{i\})c(S)} \\
&= \gamma(S) \cdot \frac{c(S) + \kappa_i(S) - c(S \cup \{i\})}{c(S \cup \{i\})}.
\end{aligned}$$

With (4.60) and (4.61) we get

$$\gamma(S \cup \{i\}) - \gamma(S) > 0.$$

Now, starting with (4.57) leads with the same arguments to

$$\gamma(S \cup \{i\}) - \gamma(S) < 0,$$

which finally proves the statement. □

An obvious candidate for a capital allocation that is suitable for performance measurement with γ and the coalition of players S is the marginal contribution allocation defined by

$$\kappa_i(S) := c(S \cup \{i\}) - c(S) = \rho\left(\sum_{j \in S} X_j + X_i\right) - \rho\left(\sum_{j \in S} X_j\right). \tag{4.62}$$

Thus a straightforward consequence of Proposition 4.2 is the following Corollary.

Corollary 4.3. *If the reward function ϑ satisfies the additivity property* (4.59) *for the coalition of players $S \in G$ and any $i \in N \setminus S$ then the marginal contribution allocation defined in* (4.62) *is suitable for performance measurement with γ and the coalition of players $S \in G$.*

Useful properties a cost allocation in this framework can have are

Efficiency: $c(N) = \sum_{i=1}^{n} \kappa_i(N)$

Symmetry: $\kappa_i(N) = \kappa_j(N)$ if $c(\{i\} \cup S) = c(\{j\} \cup S)$ for all $S \subseteq N \setminus \{i, j\}$

Dummy Player Property: $\kappa_i(N) = c(\{i\})$ if $c(\{i\} \cup S) = c(\{i\}) + c(S)$ for every subset $S \subseteq N \setminus \{i\}$.

By restricting the whole framework of this section to the set $S = N \setminus \{i\}$ the marginal contribution allocation corresponds to the so called *with-without allocation*, introduced in Merton and Perold (1993) in a risk-capital context, see (2.5). The with-without allocation has the symmetry and the dummy player property, but in general it fails to have the efficiency property. To achieve efficiency for this allocation principle a simple normalization procedure,

$$k_i = \frac{\rho(X) - \rho(X - X_i)}{\sum_{j=1}^{n} \rho(X) - \rho(X - X_j)} \rho(X),$$

can be performed, but this is only possible if $\sum_{j=1}^{n} \rho(X) - \rho(X - X_j)$ is nonzero.

The formulation of suitability for performance measurement in this section has a specific advantage. We have shown that there exists an allocation principle that is suitable for performance measurement with a performance measure γ according to Definition 4.1. This allocation principle is the with-without allocation known from Merton and Perold (1993) and Matten (1996). Therefore, if one accepts Definition 4.1 as appropriate for a specific problem-setting the advantage of this allocation, although in general it fails to have the efficiency property, is that it does not require the cost function (accordingly the risk measure with the definition in (4.54)) to have any specific properties. This allocation principle exists whether the underlying risk measure is convex, continuous, Gâteaux-differentiable or not.

The framework of Subsection 3.2 corresponds, in contrast to this section, to a fractional players framework in game theory (see Aumann and Shapley (1974)) and thus the results obtained in Subsection 3.2 are easily transferable to a game theoretic setting by using a fractional cost and reward function, ie,

$$\vartheta(u) = \theta\left(\sum_{i=1}^{n} u_i X_i\right), \quad u \in U \subset \mathbb{R}^n$$

$$c(u) = \rho\left(\sum_{i=1}^{n} u_i X_i\right), \quad u \in U \subset \mathbb{R}^n.$$

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