Reward-Risk Ratios

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Abstract

We introduce three new families of reward-risk ratios, study their properties and compare them to existing examples. All ratios in the three families are monotonic and quasi-concave, which means that they prefer more to less and encourage diversification. Members of the second family are also scale-invariant. The third family is a subset of the second one, and all its members only depend on the distribution of a return. In the second part of the paper we provide an overview of existing reward-risk ratios and discuss their properties. For instance, we show that, like the Sharpe ratio, every reward-deviation ratio violates the monotonicity property.

Keywords Reward measures, risk measures, monotonicity, quasi-concavity, scale-invariance, distribution based.

1 Introduction

Reward-risk ratios (RRRs) are widely used as performance measures in financial decision making. But not all of the existing examples have good structural properties. For instance, it is well known that the Sharpe ratio is not monotonic; see e.g. Aumann and Serrano (2008). This can lead to the situation that an investment is preferred to another one that has a higher return

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in every possible state of the world. In this paper we take the stance that every performance measure should be at least monotonic and quasi-concave. Monotonicity means that more is better than less. Quasi-concavity leads to preferences that value averages higher than extremes and therefore encourages diversification. The Sharpe ratio, like many other existing reward-risk ratios, is scale-invariant and only depends on the distribution of a return. Scale-invariance sometimes simplifies computations, but it is not founded on decision theoretic principles. That a performance measure only depends on the distribution of a return makes sense if the underlying probability measure is known. But in many real-world situations, the probabilities of uncertain events are not known exactly. Accordingly, many modern classes of preference relations do not just depend on the distributions coming from a single probability measure; see e.g. Klibanoff et al. (2005), Maccheroni et al. (2006), or Drapeau and Kupper (2013).

In this paper we introduce three families of reward-risk ratios with good structural properties and compare them to examples from the literature. The first family consists of robust reward-risk ratios that take into account expectations under different probability measures. All its members are monotonic and quasi-concave, but not necessarily scale-invariant since their denominators only depend on possible realizations of returns below a given threshold. In general they also do not depend only on the distribution of a return under a single probability measure. The second family is similar to the first one, but its members have positively homogeneous risk components. As a consequence, all reward-risk ratios in the second family are monotonic, quasi-concave and scale-invariant. The third family consists of ratios of distorted expectations. It is a subclass of the second family, and all its members only depend on the distribution of a return. Some examples from the literature are already contained in one of our three families. Others are not, since they violate the monotonicity or quasi-concavity condition. We round the paper out by providing an overview of the most common existing reward-risk ratios and a discussion of their properties. In particular, we show that mean-deviation ratios, which contain the Sharpe ratio as a special case, cannot be monotonic.

2 Definitions and preliminaries

Let $\mathcal{X}$ be a convex set of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ like, for instance, the space of p-integrable random variables L^p for some $1 \leq p < \infty$, or all essentially bounded random variables L^∞. As usual, we identify two random variables if they agree $\mathbb{P}$-almost surely and understand inequalities between them in the $\mathbb{P}$-almost sure sense. For example, $X \geq Y$ means $\mathbb{P}[X \geq Y] = 1$. By $\mathbb{E}$ we denote the expectation with respect to $\mathbb{P}$. If we take expectation under a different probability measure $\mathbb{Q}$, we will write $\mathbb{E}_\mathbb{Q}$.

An element $X \in \mathcal{X}$ is meant to be a general financial return over a time interval of length $T \in \mathbb{R}_+$. For example, it can be of the form

$$V_T, \quad \frac{V_T}{V_0}, \quad V_T - V_0 \quad \text{or} \quad \frac{V_T - V_0}{V_0},$$

where V_0 and V_T are the initial and final value of an investment portfolio. Or more generally, if B_0 and B_T are the initial and final value of a benchmark instrument, X could be any of the following:

$$\frac{V_T}{B_T}, \quad \frac{V_T}{V_0}\frac{B_0}{B_T}, \quad V_T - B_T, \quad V_T - V_0 - (B_T - B_0), \quad \frac{V_T - V_0}{B_T - B_0},$$
$$\frac{V_T - B_T}{B_T}, \quad \frac{V_T - V_0}{V_0} - \frac{B_T - B_0}{B_0}, \quad \frac{V_T - V_0}{V_0}\frac{B_0}{B_T - B_0}.$$

The purpose of this paper is to study reward-risk ratios (RRR) of the form

$$\alpha(X) = \frac{\theta(X)^+}{\rho(X)^+} \tag{2.1}$$

for a reward measure $\theta : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ and a risk measure $\rho : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$. x^+ denotes the positive part $\max\{x, 0\}$. Similarly, we denote by x^- the negative part $-\min\{x, 0\}$ and by $x \wedge y$ the minimum of x and y. Moreover, $0/0$ and ∞/∞ are both understood to be 0. Then (2.1) is well-defined in all possible cases.

To be able to cover a wide range of examples, we do not make any formal requirements on the functions θ and ρ. Instead, we directly focus on properties of the ratio α. Our main interest is in RRRs that satisfy the following two conditions:

(M) Monotonicity $\alpha(X) \geq \alpha(Y)$ for all $X, Y \in \mathcal{X}$ such that $X \geq Y$

(Q) Quasi-concavity $\alpha(\lambda X + (1-\lambda)Y) \geq \alpha(X) \wedge \alpha(Y)$ for all $X, Y \in \mathcal{X}$ and $\lambda \in \mathbb{R}$ such that $0 \leq \lambda \leq 1$.

Monotonicity is a minimal requirement that any performance measure should satisfy. It simply means that more is better than less. If it is violated, there exist $X \leq Y$ with $\mathbb{P}[X < Y] > 0$ such that X is preferred to Y. Quasi-concavity describes uncertainty aversion; see for instance, Schmeidler (1989), Cerreia-Vioglio et al. (2011), or Drapeau and Kupper (2013). If α has this property, it prefers averages to extremes and encourages diversification. If not, there exist $X, Y \in \mathcal{X}$ and a number $\lambda \in (0,1)$ such that $\alpha(\lambda X + (1-\lambda)Y) < \alpha(X) \wedge \alpha(Y)$. This can lead to concentration of risk, as is well-known from the study of Value-at-Risk; see e.g. Artzner et al. (1999).

Many RRRs that exist in the literature also satisfy

(S) Scale-invariance $\alpha(\lambda X) = \alpha(X)$ for all $X \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+ \setminus \{0\}$ such that $\lambda X \in \mathcal{X}$

(D) Distribution-based $\alpha(X)$ only depends on the distribution of X under $\mathbb{P}$.

If α satisfies (S), it does not depend on the size of X. This sometimes helps in numerical evaluations, but there is no economic or decision theoretic reason why α should have this property. A characterization of performance measures that are monotonic, quasi-concave and scale-invariant was given by Cherny and Madan (2009). Property (D) is natural if there is good reason to believe that $\mathbb{P}$ is the true underlying probability measure. A common approach in risk measurement is to work with the empirical distribution coming from observed data; see e.g. Beutner and Zähle (2010), Pflug and Wozabal (2010) or Belomestny and Krätschmer (2012). However, in many financial applications, the probabilities of uncertain events are not known precisely. If, for instance, the parameters in a model cannot be estimated with high accuracy, it might be better to take a robust approach and consider a whole family of distributions that the model generates with different parameter values. In such a situation, it makes no sense to require that α satisfies (D).

The following result gives sufficient conditions on the functions θ and ρ for α to fulfill (M), (Q), (S) or (D).

Proposition 2.1 *Let α be of the form* (2.1)*.*

1. *If $\theta(X) \geq \theta(Y)$ and $\rho(X) \leq \rho(Y)$ for all $X, Y \in \mathcal{X}$ such that $X \geq Y$, then α satisfies the monotonicity property* (M)*.*
2. *If θ is concave and ρ convex, then α satisfies the quasi-concavity property* (Q)*.*
3. *If $\rho(\lambda X) = \lambda \rho(X)$ and $\theta(\lambda X) = \lambda \theta(X)$ for all $X \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+ \setminus \{0\}$ such that $\lambda X \in \mathcal{X}$, then α satisfies the scale-invariance property* (S)*.*
4. *If θ and ρ satisfy the distibution-based property* (D)*, then so does α.*

Proof. Statements 1, 3 and 4 are obvious. 2 is readily proved as follows: Let $X, Y \in \mathcal{X}$ and $\lambda \in (0,1)$. If $\alpha(X) \wedge \alpha(Y) = 0$, one has $\alpha(\lambda X + (1-\lambda)Y) \geq 0 = \alpha(X) \wedge \alpha(Y)$. If $\alpha(X) \wedge \alpha(Y) > 0$, then $\theta(X) \wedge \theta(Y) > 0$. Concavity of θ and convexity of ρ give

$$\theta(\lambda X + (1-\lambda)Y) \geq \lambda \alpha(X) \rho(X)^+ + (1-\lambda)\alpha(Y)\rho(Y)^+ \geq (\alpha(X) \wedge \alpha(Y))\rho(\lambda X + (1-\lambda)Y)^+.$$

□

3 Three general families of RRRs

In this section we introduce three general families of RRRs. All members of the first family satisfy the monotonicity property (M) and the quasi-concavity property (Q), but are not necessarily scale-invariant or distribution-based. Members of the second one fulfill (M), (Q), (S) and those of the third one (M), (Q), (S), (D).

3.1 Robust RRRs

Let $\mathcal{P}$ and $\mathcal{Q}$ be two non-empty sets of probability measures that are absolutely continuous with respect to $\mathbb{P}$. Set

$$\theta(X) = \inf_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}[X] \quad \text{and} \quad \rho(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}\left[((m-X)^+)^p\right]\right)^{\beta/p}$$

for $m \in \mathbb{R}$, $p, \beta \geq 1$, and choose the domain $\mathcal{X}$ so that all expectations are well-defined and finite. This is always the case for $\mathcal{X} = L^\infty$. But in many examples $\mathcal{X}$ can be taken to be larger. Then

$\theta : \mathcal{X} \to \mathbb{R} \cup \{-\infty\}$ is a concave function satisfying (M) and (S). Moreover, $\rho : \mathcal{X} \to \mathbb{R}_+ \cup \{+\infty\}$ is convex and $\rho(X) \le \rho(Y)$ for $X \ge Y$. So it follows from Proposition 2.1 that the robust RRR

$$\alpha(X) = \frac{\inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}[X]^+}{\sup_{\mathbb{Q}\in\mathcal{Q}} (\mathbb{E}_{\mathbb{Q}}[((m-X)^+)^p])^{\beta/p}} \tag{3.1}$$

has the properties (M) and (Q). In the following special cases α has more structure:

a) If $\mathcal{P} = \mathcal{Q} = \{\mathbb{P}\}$, then θ and ρ satisfy (D). So α fulfills (M), (Q), (D).

b) If $m = 0$ and $\beta = 1$, ρ satisfies $\rho(\lambda X) = \lambda\rho(X)$ for all $X \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+$. Therefore, α fulfills (M), (Q), (S).

In the intersection of a) and b) lies the **Sortino–Satchell ratio** $\mathbb{E}[X]^+ / \|X^-\|_p$; see Sortino and Satchell (2001). It satisfies all four properties (M), (Q), (S), (D). For $p = 1$ one obtains the **Gains-Loss ratio** $\mathbb{E}[X]^+ / \mathbb{E}[X^-]$ of Bernardo and Ledoit (2000). Class a) extends the Sortino–Satchell ratio by generalizing the denominator. A general $m \in \mathbb{R}$ permits to shift the reference level for outcomes of X, whereas a $\beta > 1$ generates a convex distortion of the risk component.

Going beyond the class a) allows to introduce ambiguity through the sets $\mathcal{P}$ and $\mathcal{Q}$. For instance, the elements of $\mathcal{P}$ could describe the beliefs of different traders, and $\mathcal{Q}$ could consist of different stress scenarios. Enlarging $\mathcal{P}$ and $\mathcal{Q}$ increases the ambiguity aversion and makes α more conservative. A simple family of robust RRRs parametrized by $\mathcal{P}$ and $\mathcal{Q}$ is the following subclass of b):

$$\alpha(X) = \frac{\inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}[X]^+}{\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[X^-]}. \tag{3.2}$$

3.2 Robust expectations ratios

Note that members of the family (3.2) can be written in the form

$$\alpha(X) = \frac{\inf_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}[X]^+}{\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-X]^+} \tag{3.3}$$

by modifying the measures $\mathbb{Q} \in \mathcal{Q}$ so that their support is in $\{X \le 0\}$. But since in (3.3) the denominator can depend on the positive tail of X, (3.3) is more general than (3.2) and not fully covered by the class of robust RRRs (3.1). Like all members of (3.2), RRRs of the form (3.3) satisfy (M), (Q) and (S).

3.3 Distortion RRRs

We now discuss a subclass of (3.3) parametrized by two distortion functions. A distortion function is a non-decreasing function $\varphi : [0,1] \to [0,1]$ satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$. It induces the distorted probability $\mathbb{P}_\varphi(A) := \varphi \circ \mathbb{P}[A]$, which can be used to define the distorted expectation $\mathbb{E}_\varphi$ as a Choquet integral:

$$\mathbb{E}_\varphi[X] := \int_0^\infty \mathbb{P}_\varphi[X > t]dt + \int_{-\infty}^0 (\mathbb{P}_\varphi[X > t] - 1)dt, \quad X \in L^\infty.$$

It is well-known that $\mathbb{E}_\varphi$ has the following properties:

(i) $\mathbb{E}_\varphi[X] \geq \mathbb{E}_\varphi[Y]$ for $X \geq Y$

(ii) $\mathbb{E}_\varphi[\lambda X] = \lambda \mathbb{E}_\varphi[X]$ for all $\lambda \in \mathbb{R}_+$

(iii) $\mathbb{E}_\varphi[X + m] = \mathbb{E}_\varphi[X] + m$ for $m \in \mathbb{R}$

(iv) $\mathbb{E}_\varphi$ only depends on the distribution of X

(v) If φ is convex, then $\mathbb{E}_\varphi$ is concave and

$$\mathbb{E}_\varphi[X] = \inf_{\mathbb{Q} \in \mathcal{P}_\varphi} \mathbb{E}_\mathbb{Q}[X], \quad \text{where } \mathcal{P}_\varphi := \{\mathbb{Q} \ll \mathbb{P} : \mathbb{Q}[A] \leq \mathbb{P}_\varphi[A] \text{ for all } A \in \mathcal{F}\}$$

(vi) If φ is concave, then $\mathbb{E}_\varphi$ is convex and

$$\mathbb{E}_\varphi[X] = \sup_{\mathbb{Q} \in \mathcal{Q}_\varphi} \mathbb{E}_\mathbb{Q}[X], \quad \text{where } \mathcal{Q}_\varphi := \{\mathbb{Q} \ll \mathbb{P} : \mathbb{Q}[A] \geq \mathbb{P}_\varphi[A] \text{ for all } A \in \mathcal{F}\};$$

see e.g., Schmeidler (1985), Denneberg (1997) or Delbaen (2002).

If $\theta(X) = \mathbb{E}_\varphi[X]$ for a convex distortion function φ, $\rho(X) = \mathbb{E}_\psi[-X]$ for a concave distortion function ψ, and $\mathcal{X} \supseteq L^\infty$ is chosen so that $\mathbb{E}_\varphi[X], \mathbb{E}_\psi[-X] \in \mathbb{R}$ for all $X \in \mathcal{X}$, one obtains from Proposition 2.1, that the distortion RRR

$$\alpha_{\varphi,\psi}(X) = \frac{\mathbb{E}_\varphi[X]^+}{\mathbb{E}_\psi[-X]^+}, \quad X \in \mathcal{X}, \tag{3.4}$$

has all four properties (M), (Q), (S), (D). It can be seen from (v) and (vi) that distortion RRRs are contained in the class of robust expectations ratios (3.3).

For $\varphi = id$ one obtains $\theta(X) = \mathbb{E}[X]$. Some popular choices of concave distortion functions to describe risk from the literature are the following:

a) The proportional hazard transform (Wang, 1995)

$$\psi(x) = x^{1/\gamma} \quad \text{for some } \gamma \geq 1,$$

which, via $\rho(X) = \mathbb{E}_\psi[-X]$, leads to a risk assessment that inflates the probabilities of large losses.

b) The Wang transform (Wang, 2000)

$$\psi(x) = \Phi\left[\Phi^{-1}(x) + \gamma\right] \quad \text{for some } \gamma \geq 0,$$

where Φ is the standard normal cdf. Again, ψ inflates the probabilities of large losses. For instance, if X is normally distributed with mean μ and standard deviation σ, $\mathbb{E}_\psi[-X]$ is equal to the expectation of $-Y$, where Y is normally distributed with mean $\mu - \sigma\gamma$ and standard deviation σ.

c) The MINVAR distortion function

$$\psi(x) = 1 - (1-x)^{1+\gamma} \quad \text{for some } \gamma \geq 0.$$

Cherny and Madan (2009) showed that in the case where γ is an integer, $\mathbb{E}_\psi[-X]$ is equal to the expectation of $-Y$, where $Y = \min\{X_1, \ldots, X_{\gamma+1}\}$ and $X_1, \ldots, X_{\gamma+1}$ are independent copies of X.

d) The MINMAXVAR distortion function

$$\psi(x) = 1 - (1 - x^{1/(1+\gamma)})^{1+\gamma} \quad \text{for some} \quad \gamma \geq 0.$$

It is shown in Cherny and Madan (2009) that for integer γ, $\mathbb{E}_\psi[-X]$ is equal to the expectation of $-Y$, where $Y = \min\{Z_1, \ldots, Z_{\gamma+1}\}$ and $Z_1, \ldots, Z_{\gamma+1}$ are independent random variables such that $\max\{Z_1, \ldots, Z_{\gamma+1}\}$ has the same distribution as X.

In addition, several of the examples of the next section can be written as distortion RRRs; see Subsection 4.4 below. In particular, Value-at-Risk and Average-Value-at-Risk can be represented as distortion risk measures. For further information on distortion risk measures in portfolio optimization we refer to Sereda et al. (2010).

4 More examples

In this section we study properties of some reward-risk ratios from the literature and discuss extensions.

4.1 Mean-deviation ratios

The archetypical RRR is the **Sharpe ratio** $\mathrm{SR}(X) := \mathbb{E}[X]^+ / \sigma(X)$ (Sharpe, 1966), where $\sigma(X)$ denotes the standard deviation $\sigma(X) := \|X - \mathbb{E}[X]\|_2$. Generalizing the standard deviation, Rockafellar et al. (2006) introduced the class of deviation measures as all functions $\mathcal{D} : L^2 \to \mathbb{R}_+ \cup \{+\infty\}$ satisfying

(D1) $\mathcal{D}(X + m) = \mathcal{D}(X)$ for all $X \in L^2$ and $m \in \mathbb{R}$

(D2) $\mathcal{D}(0) = 0$, and $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all $X \in L^2$ and all $\lambda \in \mathbb{R}_+ \setminus \{0\}$

(D3) $\mathcal{D}(X+Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ for all $X, Y \in L^2$

(D4) $\mathcal{D}(X) > 0$ for all nonconstant $X \in L^2$, and $\mathcal{D}(X) = 0$ for contant $X \in L^2$.

Every deviation measure $\mathcal{D}$ induces a **Mean-Deviation ratio** of the form $\alpha_{\mathcal{D}} := \mathbb{E}[X]^+ / \mathcal{D}(X)$. It follows from Proposition 2.1 that each mean-deviation ratio satisfies (Q) and (S). On the other hand, one has the following:

Proposition 4.1 *Fix $p \in [1, \infty]$ and let $\theta : L^p \to \mathbb{R} \cup \{\pm\infty\}$ be a function such that $\theta(X+m) = \theta(X) + m$ and $\theta(\lambda X) = \lambda\theta(X)$ for all $X \in L^p$, $m \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+ \setminus \{0\}$. If $\mathcal{D} : L^p \to \mathbb{R}_+ \cup \{+\infty\}$ satisfies* (D1)–(D2) *and there exists a $Z \in L^p$ such that $Z \geq 0$, $\theta(Z) \in \mathbb{R}_+$ and $\mathcal{D}(Z) \in \mathbb{R}_+ \setminus \{0\}$, then the corresponding RRR $\alpha(X) = \theta(X)^+ / \mathcal{D}(X)$ violates the monotonicity property* (M).

Proof. Let $Z \in L^p$ such that $Z \geq 0$, $\theta(Z) \in \mathbb{R}_+$ and $\mathcal{D}(Z) \in \mathbb{R}_+ \setminus \{0\}$. Choose constants $a, b, c > 0$ such that $a > c$, $b \geq 1$ and $bc > a$. Define $X = a + bZ$ and $Y = c + Z$. Then $X > Y$, and at the same time,

$$\alpha(X) = \frac{a}{b} \frac{1}{\mathcal{D}(Z)} + \alpha(Z) < \frac{c + \theta(Z)}{\mathcal{D}(Z)} = \alpha(Y).$$

This shows that α violates (M). □

As a special case of Proposition 4.1 one obtains that mean-deviation ratios do not satisfy the monotonicity property (M).

Corollary 4.2 *Fix $p \in [1, \infty]$ and let $\mathcal{D} : L^p \to \mathbb{R}_+ \cup \{+\infty\}$ be a function with the properties* (D1)–(D2). *If there exists a $Z \in L^p$ such that $Z \geq 0$ and $\mathcal{D}(Z) \in \mathbb{R}_+ \setminus \{0\}$, then the ratio $\mathbb{E}[X]^+ / \mathcal{D}(X)$ violates the monotonicity condition* (M).

One can modify the Sharpe ratio by replacing the standard deviation by a p-deviation $\sigma_p(X) := \|\mathbb{E}[X] - X\|_p$ or a p-semi-deviation $\sigma_p^-(X) := \|(\mathbb{E}[X] - X)^+\|_p$, $p \geq 1$. By Proposition 2.1, the **Mean-p-Deviation ratio** $\mathbb{E}[X]^+ / \sigma_p(X)$ as well as the **Mean-p-Semi-Deviation ratio** $\mathbb{E}[X]^+ / \sigma_p^-(X)$ fulfill (Q), (S), (D). However, since σ_p and σ_p^- satisfy (D1)–(D4), one obtains from Corollary 4.2 that both ratios violate the monotonicity condition (M). Special cases are the **Mean-Standard-Semi-Deviation ratio** $\mathbb{E}[X]^+ / \|(X - \mathbb{E}[X])^+\|_2$ (see Martin and McCann, 1989), the **Mean-Absolute-Deviation ratio** $\mathbb{E}[X]^+ / \mathbb{E}[|X - \mathbb{E}[X]|]$ (see Konno and Yamazaki, 1991) and the **Mean-Absolute-Semi-Devation ratio** $\mathbb{E}[X]^+ / \mathbb{E}[(X - \mathbb{E}[X])^+]$. Further variations of the Sharpe ratio include the **Skewness Adjusted Sharpe ratio**

$$\mathrm{SASR}_b(X) := \mathrm{SR}(X) \sqrt{1 + \frac{b}{3} \mathbb{E}\left[\left(\frac{X - \mathbb{E}[X]}{\sigma(X)}\right)^3\right]},$$

the **Adjusted for Skewness Sharpe ratio**

$$\mathrm{ASSR}_b(X) := \mathrm{SR}(X) \sqrt{1 + \frac{b}{3} \mathbb{E}\left[\left(\frac{X - \mathbb{E}[X]}{\sigma(X)}\right)^3\right] \mathrm{SR}(X)}$$

(see Zakamouline and Koekebakker, 2008), and the **Black–Treynor ratio**

$$\mathrm{BTR}(X) = \frac{\mathbb{E}[X]^+}{\mathrm{Cov(X,B)}^+/\mathrm{Var(B)}}$$

for a benchmark instrument B (see Treynor and Black, 1973). It follows from Proposition 4.1 that SASR and BTR do not satisfy the monotonicity condition (M). ASSR does not exactly fulfill the assumptions of Proposition 4.1. But it can be shown that it violates (M) as well. SASR and ASSR satisfy (S) and (D), but both violate (Q). Indeed, if (X, Y) is a pair of random variables with distribution

$$(X,Y) = \begin{cases} (4,2) & \text{with probability } 0.5 \\ (-1,4) & \text{with probability } 0.4 \\ (-2,-4) & \text{with probability } 0.1 \end{cases},$$

one has $\mathrm{SASR}_1(X) \approx 0.53$, $\mathrm{SASR}_1(Y) \approx 0.60$, but

$$\mathrm{SASR}_1(0.3X + 0.7Y) \approx 0.37 < 0.53 \approx \min\{\mathrm{SASR}_1(X), \mathrm{SASR}_1(Y)\}.$$

The same example also shows that ASSR does not satisfy (Q). By Proposition 2.1, BTR satisfies (Q) and (S). However, since it depends on the joint distribution of X and B, it does in general not fulfill (D).

4.2 Mean-Risk ratios with monetary risk measures

A mapping $\rho : L^p \to (-\infty, \infty]$ is a monetary risk measure if it has the properties (R1)–(R2) below. If it satisfies (R1)–(R3), it is called a convex risk measure. If it fulfills (R1)–(R4), it is a coherent risk measure; see Artzner et al. (1999) and Föllmer and Schied (2004).

(R1) $\rho(X) \geq \rho(Y)$ for $X \leq Y$,

(R2) $\rho(X + m) = \rho(X) - m$ for all $X \in \mathcal{X}$ and $m \in \mathbb{R}$,

(R3) $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for all $X, Y \in \mathcal{X}$ and $\lambda(0, 1)$,

(R4) $\rho(\lambda X) = \lambda\rho(X)$ for all $X \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+$.

4.2.1 Value-at-Risk ratio

The **Value-at-Risk ratio** of Favre and Galeano (2002) (also called **RoVaR**) is given by

$$\mathrm{VaRR}_\gamma(X) := \frac{\mathbb{E}[X]^+}{\mathrm{VaR}_\gamma(X)^+},$$

where $\mathrm{VaR}_\gamma(X) = \inf\{m \in \mathbb{R} : \mathbb{P}[X + m < 0] \leq \gamma\}$ is the **Value-at-Risk** at the level $\gamma \in (0, 1)$. It is well-known that VaR_γ satisfies (D), (R1), (R2), (R4) but not (R3); see e.g. Artzner et al. (1999). It follows that VaRR_γ fulfills (M), (S), (D). But it violates the quasi-concavity property (Q). This can be seen by considering two i.i.d. random variables X, Y with distribution

$$X = \begin{cases} -30 & \text{with probability } 0.03 \\ -10 & \text{with probability } 0.03 \\ 5 & \text{with probability } 0.94 \end{cases}.$$

Then $\mathbb{E}[X] = \mathbb{E}[Y] = 3.5$, and for $\gamma = 0.05$, $\mathrm{VaR}_\gamma(X) = \mathrm{VaR}_\gamma(Y) = 10$, $\mathrm{VaR}_\gamma(X+Y) = 25$. Since VaRR_γ is scale-invariant, one has

$$\mathrm{VaRR}_\gamma\left(\frac{X+Y}{2}\right) = \mathrm{VaRR}_\gamma(X+Y) = \frac{7}{25} < \frac{7}{20} = \mathrm{VaRR}_\gamma(X) \wedge \mathrm{VaRR}_\gamma(Y),$$

which shows that VaRR_γ is not quasi-concave.

4.2.2 Mean-Risk ratios with convex risk measures

A mean-risk ratio with a convex risk measure ρ in the denominator that is not coherent satisfies (M) and (Q) but not (S). For instance, if ρ is an entropic risk measure $\gamma^{-1} \log \mathbb{E}[\exp(-\gamma X)]$ for a constant $\gamma > 0$, one obtains the **Mean-Entropic ratio**

$$\mathrm{MER}_\gamma(X) := \frac{\gamma \mathbb{E}[X]^+}{\log \mathbb{E}[\exp(-\gamma X)]^+}.$$

More generally, if ρ is a distortion-exponential risk measure $\gamma^{-1} \log \mathbb{E}_\psi[\exp(-\gamma X)]$ for a constant $\gamma > 0$ and a concave distortion function ψ (see Tsanakas, 2009), one gets the **Mean-Distortion-Entropic ratio**

$$\mathrm{MDER}_{\gamma,\psi}(X) := \frac{\gamma \mathbb{E}[X]^+}{\log \mathbb{E}_\psi[\exp(-\gamma X)]^+}.$$

4.2.3 Mean-Risk ratios with coherent risk measures

If ρ is a coherent risk measure, the mean-risk ratio $\mathbb{E}[X]^+ / \rho(X)^+$ satisfies (M), (Q), (S). The following examples are from the literature. They all satisfy (D).

The **Stable Tail Adjusted Return ratio** (see Martin et al., 2003) is given by

$$\mathrm{STARR}_\gamma(X) := \frac{\mathbb{E}[X]^+}{\mathrm{AVaR}_\gamma(X)^+},$$

where $\mathrm{AVaR}_\gamma(X) := \gamma^{-1} \int_0^\gamma \mathrm{VaR}_u(X) du$ is the Average-Value-at-Risk at the level $\gamma \in (0, 1]$. Since AVaR_γ dominates VaR_γ, STARR_γ is below VaRR_γ.

The **MiniMax ratio** (see Young, 1998) is defined as

$$\mathrm{MMR}(X) := \frac{\mathbb{E}[X]^+}{\|X^-\|_\infty}.$$

The **Gini ratio** (see Shalit and Yitzhaki, 1994) is

$$\mathrm{GR}_\gamma(X) := \frac{\mathbb{E}[X]^+}{(\Gamma_X(\gamma) - \mathbb{E}[X])^+},$$

where $\Gamma_X(\gamma) := \mathbb{E}[X] - \gamma \int_0^1 (1-u)^{\gamma-1} F_X^{-1}(u) du$. As mentioned in Ortobelli et al. (2006), $\Gamma_X(\gamma) - \mathbb{E}[X]$ is a coherent risk measure for every $\gamma > 1$.

4.3 More non-quasi-concave reward-risk ratios

The **Farinelli–Tibiletti ratio** (Farinelli and Tibiletti, 2008) is given by

$$\mathrm{FTR}(X) := \frac{\|(X-m)^+\|_p}{\|(n-X)^+\|_q},$$

where $m, n \in \mathbb{R}$ and $p, q > 0$. In general it only satisfies (M) and (D). If $m = n = 0$ it also fulfills (S), but since the numerator is not concave, it violates (Q). Indeed, consider a pair of random variables (X, Y) with distribution

$$(X,Y) = \begin{cases} (-10, 1) & \text{with probability } 0.5 \\ (1, 3) & \text{with probability } 0.2 \\ (2, -6) & \text{with probability } 0.3 \end{cases}$$

and set $m = n = p = q = 1$. Then $\mathrm{FTR}(X) = 3/55$, $\mathrm{FTR}(Y) = 4/21$ and for $\lambda = 3/4$, one has

$$\mathrm{FTR}(\lambda X + (1-\lambda)Y) = \frac{4}{177} < \frac{3}{55} = \mathrm{FTR}(X) \wedge \mathrm{FTR}(Y).$$

The **Skewness-Kurtosis ratio** (Watanabe, 2006) is given by

$$\mathrm{SKR}(X) = \frac{\mathbb{E}\left[\phi(X)^3\right]^+}{\mathbb{E}\left[\phi(X)^4\right]} \quad \text{for } \phi(X) = \frac{X - \mathbb{E}[X]}{\sigma(X)}.$$

It obviously satisfies (S) and (D). But it violates (M) and (Q), which can be seen from the following two examples: First, assume that

$$(X,Y) = \begin{cases} (3, 1) & \text{with probability } 0.25 \\ (2, 0) & \text{with probability } 0.5 \\ (1, 0) & \text{with probability } 0.25 \end{cases}.$$

Then $X > Y$. But $\mathrm{SKR}(X) = 0 < \mathrm{SKR}(Y) = 2\sqrt{3}/7$, which shows that (M) is violated. Furthermore, if

$$(X,Y) = \begin{cases} (-10, 1) & \text{with probability } 0.5 \\ (1, 3) & \text{with probability } 0.2 \\ (2, -1) & \text{with probability } 0.3 \end{cases},$$

then $\mathrm{SKR}(X) \approx 0.0104 < 0.0686 \approx \mathrm{SKR}(Y)$, and for $\lambda = 0.7$ one has $\mathrm{SKR}(\lambda X + (1-\lambda)Y) \approx 0.0056 < 0.0104 \approx \mathrm{SKR}(X) \wedge \mathrm{SKR}(Y)$. So SKR does not satisfy (Q).

The **Rachev ratio** and the **Generalized Rachev ratio** (Biglova et al., 2004) are given by

$$\mathrm{RR}_{(\beta,\gamma)}(X) := \frac{\mathrm{AVaR}_\beta(-X)}{\mathrm{AVaR}_\gamma(X)} \quad \text{and} \quad \mathrm{GRR}_{(\beta,\gamma,\delta,\varepsilon)}(X) := \frac{\mathrm{AVaR}_{(\beta,\gamma)}(-X)}{\mathrm{AVaR}_{(\delta,\varepsilon)}(X)}$$

for $\mathrm{AVaR}_{(\beta,\gamma)}(X) := \beta^{-1} \int_0^\beta [\max(-F_X^{-1}(u), 0)]^\gamma du$. A variant of GRR is the **Modified Generalized Rachev ratio** (Stoyanov et al., 2007)

$$\mathrm{MGRR}_{(\beta,\gamma,\delta,\varepsilon)}(X) := \frac{\mathrm{AVaR}_{(\beta,\gamma)}(-X)^{1/\gamma}}{\mathrm{AVaR}_{(\delta,\varepsilon)}(X)^{1/\varepsilon}}. \tag{4.1}$$

While RR and MGRR fulfill (M), (S) and (D), GRR only satisfies (M) and (D). Moreover, all three ratios have a non-concave numerator and therefore violate (Q). For instance, consider $\alpha = \mathrm{RR}_{(\beta,\gamma)}$ with $\beta = \gamma = 0.05$. It is well-known that $\mathrm{AVaR}_\beta(X)$ is equal to $\beta^{-1}\mathbb{E}[(q-X)^+] - q$, where q is a β-quantile of X; see Föllmer and Schied (2004). So if the pair (X, Y) is distributed like

$$(X,Y) = \begin{cases} (-500, -1000) & \text{with probability } 0.02 \\ (4, -5) & \text{with probability } 0.03 \\ (1, 6) & \text{with probability } 0.95 \end{cases},$$

one obtains $\alpha(X) = 14/997$ and $\alpha(Y) = 6/403$. On the other hand,

$$RR_{(\beta,\gamma)}\left(\frac{X+Y}{2}\right) = RR_{(\beta,\gamma)}(X+Y) = \frac{5}{429} < \frac{14}{997} = RR_{(\beta,\gamma)}(X) \wedge RR_{(\beta,\gamma)}(Y),$$

which shows that RR, GRR and MGRR all violate (Q).

4.4 Representation as distortion RRR

Several of the RRRs in Section 4 can be written as distortion RRRs. As observed in Section 3, the mean can be expressed as $\mathbb{E}_\varphi$ for $\varphi = id$. Furthermore, one can represent $\mathrm{VaR}_\gamma(X)$ as $\mathbb{E}_{\psi_\gamma}[-X]$ for

$$\psi_\gamma(x) = \begin{cases} 0 & \text{if } 0 \le x < \gamma \\ 1 & \text{if } \gamma \le x \le 1 \end{cases},$$

$\mathrm{AVaR}_\gamma(X)$ as $\mathbb{E}_{\psi_\gamma}[-X]$ for

$$\psi_\gamma(x) = \begin{cases} x/\gamma & \text{if } 0 \le x \le \gamma \\ 1 & \text{if } \gamma \le x \le 1 \end{cases}, \tag{4.2}$$

and the Gini-measure $\Gamma_X(\gamma) - \mathbb{E}[X]$ as $\mathbb{E}_{\psi_\gamma}[-X]$ for

$$\psi_\gamma(x) = (1-x)^\gamma, \quad \gamma > 1.$$

The modified generalized Rachev ratio (4.1) can be represented as

$$\mathrm{MGRR}_{(\beta,\gamma,\delta,\varepsilon)}(X) = \frac{\mathbb{E}_{\psi_\beta}[(X^+)^\gamma]^{\frac{1}{\gamma}}}{\mathbb{E}_{\psi_\delta}[((-X)^+)^\varepsilon]^{\frac{1}{\varepsilon}}},$$

with ψ_β and ψ_δ as in (4.2).

4.5 Summary of properties

The following table lists properties of the RRRs discussed in this paper.

Ratio	(M)	(Q)	(S)	(D)
Robust RRR	x	x		
Robust Expectations Ratio	x	x	x	
Distortion RRR	x	x	x	x
Sortino–Satchel ratio	x	x	x	x
Gains-Loss ratio	x	x	x	x
Mean-Deviation ratio		x	x	
Sharpe ratio		x	x	x
Mean-p-Deviation ratio		x	x	x
Mean-p-Semi-Deviation ratio		x	x	x
Skewness Adjusted Sharpe ratio			x	x
Adjusted for Skewness Sharpe ratio			x	x
Black–Treynor ratio		x	x	
Value-at-Risk ratio	x		x	x
Mean-Risk ratio with a convex risk measure	x	x		
Mean-Entropic ratio	x	x		x
Mean-Distortion-Entropic ratio	x	x		x
Mean-Risk ratio with a coherent risk measure	x	x	x	
STAR ratio	x	x	x	x
MiniMax ratio	x	x	x	x
Gini ratio	x	x	x	x
Farinelli–Tibiletti ratio	x		x	x
Skewness-Kurtosis ratio			x	x
Rachev ratio	x		x	x
Generalized Rachev ratio	x			x
Modified Generalized Rachev ratio	x		x	x

Table 1: Properties of reward-risk ratios

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