Feynman-Kac for functional jump diffusions with an application to Credit Value Adjustment

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Abstract

We provide a proof for the functional Feynman-Kac theorem for jump diffusions with path-dependent coefficients. To obtain this result we first study the existence and uniqueness of solutions to functional jump diffusions and derive a useful bound for the solution. We apply our results to the problem of Credit Value Adjustment (CVA) in a bilateral counterparty risk framework. We derive the corresponding functional CVA-PIDE and extend existing results on CVA to a setting which enables the pricing of path-dependent derivatives.

Keywords Functional Feynman-Kac Theorem, functional Itô formula, functional jump diffusion, path-dependent coefficients, Credit Value Adjustment, bilateral counterparty risk, path-dependent derivatives, Asian option.

1 Introduction

In a very insightful paper Dupire (2009) introduced a functional Itô formula for continuous processes which generalizes the classical one. This approach was extended to a functional Itô formula for càdlàg semimartingales by Cont and Fournié (2010) and Levental et al. (2013). In our work we will first derive an existence and uniqueness result for jump diffusions with path-dependent coefficients. This proof will directly yield a bound on the solution, which will then be necessary for our proof of the functional Feynman-Kac theorem for path-dependent jump diffusions. In the following these results will enable us to study Credit Value Adjustment (CVA) in a setting which allows for market models with path-dependent coefficients of the underlying SDEs. Furthermore, by introducing jumps of the asset process S we will derive our results in an incomplete market model where the jump risk of S can not be hedged. In this generalized setting we will derive the appropriate CVA-PIDE and our functional Feynman-Kac theorem will yield the corresponding representation of the solution to this PIDE. Finally this will enable the CVA-valuation of path-dependent derivatives, which was not possible in the previous approaches to CVA and will generalize the market model to include jumps of the asset price process. By removing the jumps of the asset price process we get a complete market model where

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all risks can be hedged and state the corresponding functional CVA-PDE. This directly generalizes the approach of Burgard and Kjaer (2011) by allowing for path-dependent derivatives.

2 Preliminaries

In this section we will follow the notation of Levental et al. (2013) and collect definitions and notations that will be needed in the following sections. We will first introduce the notation of paths of stochastic processes, the space of càdlàg functions and directional derivatives of functionals on this space. Then we will state the functional Itô formula for càdlàg semimartingales from Levental et al. (2013). Let $X : [0,T]\times\Omega \to \mathbb{R}^d$ be an $\mathbb{R}^d$-valued, adapted càdlàg semimartingale defined on the filtered probability space $(\Omega, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{F}, \mathbb{P})$. Denote by $D([0,T],\mathbb{R}^d)$ the space of all càdlàg $\mathbb{R}^d$-valued functions on $[0,T]$ endowed with the supremum metric. We define the path X_t, $t\in[0,T]$, of the stochastic process X as $X_t(s) = X(s\wedge t)$ for all $s\in[0,T]$. The path with a shift by $h\in\mathbb{R}$ in t is denoted by $X_t^h(s) = X(s\wedge t) + h\mathbb{1}_{[t,T]}(s)$, for $s\in[0,T]$.

Definition 2.1. *Let $F : D([0,T],\mathbb{R}^d) \to \mathbb{R}$ be some functional and $\{e_i\}_{i=1,\dots,d}$ the canonical basis of $\mathbb{R}^d$. For $y\in D([0,T],\mathbb{R}^d)$ we denote by $D_i^1 F(y;[t,T])$, $i=1,\dots,d$, the derivative of F at y in the direction of the $\mathbb{R}^d$-valued process $\mathbb{1}_{[t,T]}e_i$, which means*

$$D_i^1 F(y;[t,T]) = \lim_{h\to 0} \frac{F(y+h\mathbb{1}_{[t,T]}e_i) - F(y)}{h}.$$

By

$$D_{ij}^2 F(y;[t,T]) = \lim_{h\to 0} \frac{D_j^1 F(y+h\mathbb{1}_{[t,T]}e_i;[t,T]) - D_j^1 F(y;[t,T])}{h}$$

we denote the second-order derivative in the direction $\mathbb{1}_{[t,T]}e_i$ and $\mathbb{1}_{[t,T]}e_j$, for $i,j=1,\dots,d$.

These derivatives are consistent with the derivatives from Dupire (2009). For a comparison of these approaches we refer the reader to Appendix A.1 in Levental et al. (2013). Note that the derivatives are functions of $(x,t)\in D([0,T],\mathbb{R}^d)\times[0,T]$. The distance on $D([0,T],\mathbb{R}^d)\times[0,T]$ is defined by $\tilde{d}((x,t),(y,s)) = \|x-y\|_\infty + |t-s|$, for all $(x,t),(y,s)\in D([0,T],\mathbb{R}^d)\times[0,T]$, where $\|x-y\|_\infty = \sup_{0\le t\le T}\|x(t)-y(t)\|$.

In the following we will need the functional Itô formula for càdlàg semimartingales from Levental et al. (2013).

Theorem 2.2. *Let X be a d-dimensional càdlàg semimartingale. Let $F : D([0,T],\mathbb{R}^d) \to \mathbb{R}$ be a functional which is continuous and $D_i^1 F(x;[t,T])$, $D_{ij}^2 F(x;[t,T])$, $i,j=1,\dots,d$, exist and are continuous in x and t with respect to the metric $\tilde{d}$. Then*

$$\begin{aligned} F(X_t) &= F(X_0) + \sum_{i=1}^d \int_0^t D_i^1 F(X_{s-};[s,T])\, dX^i(s) + \frac{1}{2}\sum_{i,j=1}^d \int_0^t D_{ij}^2 F(X_{s-};[s,T])\, d\langle X^{i,c}(s), X^{i,c}(s)\rangle \\ &\quad + \sum_{s\le t}\left[F(X_s) - F(X_{s-}) - \sum_{i=1}^d D_i^1 F(X_{s-};[s,T])\,\Delta X^i(s)\right], \end{aligned}$$

where X^c is the continuous part of the semimartingale X and $\Delta X^i(s) = X^i(s) - X^i(s-)$.

Remark 2.3. *If we construct $\tilde{F} : D([0,T],\mathbb{R}_+\times\mathbb{R}^d) \to \mathbb{R}$ such that $\tilde{F}(x^0,x) = F(x^0(T)\wedge T, x)$ for a functional $F : [0,T]\times D([0,T],\mathbb{R}^d) \to \mathbb{R}$ it follows for $x^0\in D([0,T],\mathbb{R}_+)$ with $x^0(s) = s$ the time and path dependent Itô formula for càdlàg semimartingales*

$$\begin{aligned} F(t,X_t) &= F(0,X_0) + \int_0^t D_s F(s,X_{s-};[s,T])\, ds + \sum_{i=1}^d \int_0^t D_i^1 F(s,X_{s-};[s,T])\, dX^i(s) \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_0^t D_{ij}^2 F(s,X_{s-};[s,T])\, d\langle X^{i,c}(s), X^{i,c}(s)\rangle + \sum_{s\le t}\left[F(s,X_s) - F(s,X_{s-}) - \sum_{i=1}^d D_i^1 F(s,X_{s-};[s,T])\,\Delta X^i(s)\right], \end{aligned}$$

with $D_t F(t,X_{t-};[t,T]) := \frac{\partial}{\partial t} F(t,X_{t-})$. Additionally we also refer to the version of the functional Itô formula in Cont and Fournié (2010).

Now, we can proceed to the next section where we will derive our main theoretical results which we will subsequently apply in Section 4 to obtain the Credit Value Adjustment PIDE and the stochastic representation of the solution to this PIDE.

3 Functional Feynman-Kac theorem to strong solutions for Lévy driven semimartingales

Consider the equation

$$dX(t+s) = b(t+s, X_{(t+s)-})ds + \sigma(t+s, X_{(t+s)-})dW(s) + \int_{|z|<c} \bar{\gamma}(t+s, X_{(t+s)-}, z)\bar{\mu}(ds,dz) + \int_{|z|\geq c} \gamma(t+s, X_{(t+s)-}, z)\mu(ds,dz), \tag{3.1}$$

with initial value

$$X_t = y_t,$$

where $W = (W(t), t \in [0,T])$ is a Wiener process, $y \in D([0,T],\mathbb{R}^d)$, $t \in [0,T]$. $\mu(ds,dz)$ is an independent Poisson random measure coming from a Lévy process and $\bar{\mu}(ds,dz) = \mu(ds,dz) - ds\nu(dz)$ is the compensated jump measure with Lévy measure ν. We assume that the function

$$(t,x,z) \to (b(t,x), \sigma(t,x), \gamma(t,x,z), \bar{\gamma}(t,x,z))$$

defined on $[0,T] \times D([0,T],\mathbb{R}^d) \times \mathbb{R}^d$ is Borel measurable. Define $a(t,x_1,x_2) := \sigma(t,x_1)\sigma(t,x_2)^T$ for each $t \geq 0$, $x_1, x_2 \in D([0,T],\mathbb{R}^d)$ and consider the conditions

(C1) Lipschitz condition: For each $s > t$ there exists $K_1(s) > 0$ such that, for all $x_1, x_2 \in D([0,T],\mathbb{R}^d)$

$$|b(s,x_1) - b(s,x_2)|^2 + \|a(s,x_1,x_1) - 2a(s,x_1,x_2) + a(s,x_2,x_2)\| + \int_{|z|<c} |\bar{\gamma}(s,x_1,z) - \bar{\gamma}(s,x_2,z)|^2 \nu(dz) \leq K_1(s) \sup_{u\leq s} |x_1(u) - x_2(u)|^2$$

(C2) Growth condition: For each $s > t$ there exists $K_2(s) > 0$ such that, for all $x \in D([0,T],\mathbb{R}^d)$,

$$|b(s,x)|^2 + \|a(s,x,x)\| + \int_{|z|<c} |\bar{\gamma}(s,x,z)|^2 \nu(dz) \leq K_2(s)(1 + \sup_{u\leq s} |x(u)|^2).$$

(C3) Continuity of γ: The mapping $x \to \gamma(t,x,z)$ is continuous for all $|z| \geq c$, $t > 0$ with respect to the metric $\tilde{d}$.

Furthermore, we assume $\sup_{s\in[0,T]} K_i(s) < \infty$, for $i = 1,2$.

First, we will prove that under the conditions (C1)-(C2) there exists a unique càdlàg solution to a modification of (3.1), namely

$$dX(t+s) = b(t+s, X_{(t+s)-})ds + \sigma(t+s, X_{(t+s)-})dW(s) + \int_{|z|<c} \bar{\gamma}(t+s, X_{(t+s)-}, z)\bar{\mu}(ds,dz). \tag{3.2}$$

A result on existence and uniqueness of solutions to SDEs with path-dependent coefficients already exists in Chapter 5, Theorem 7 in Protter (2004). Nevertheless, we will provide another proof for this result which uses similar techniques as the proof of Theorem 5.2.9 in Karatzas and Shreve (1991) for classical SDEs without jumps and of Theorem 6.2.9 in Applebaum (2005). These techniques will yield a useful bound on the solution to SDEs with path-dependent coefficients and jumps which will be necessary for the proof of the functional Feynman-Kac formula.

Theorem 3.1. *Suppose that the coefficients b, σ and $\bar{\gamma}$ satisfy the conditions (C1)-(C2). Then, if*

$$\sup_{s\in[0,t]} |y_t(s)|^2 < \infty,$$

the stochastic differential equation (3.2) has a unique strong solution. Moreover, for every $T > 0$ there exists a constant C depending only on K_2 and T such that for all $s \in [t,T]$

$$\mathbb{E}\left[\sup_{u\in[0,s]} |X(u)|^2\right] \leq C\left(1 + \sup_{u\in[0,t]} |y_t(u)|^2\right) e^{C(s-t)}. \tag{3.3}$$

Proof. Suppose

$$\sup_{s\in[0,t]} |y_t(s)|^2 < \infty$$

holds. Define for $s \in [0,T]$ the following sequence of processes

$$\begin{aligned}
X^0(s) &= y_t(s)\mathbb{1}_{s\le t} + y_t(t)\mathbb{1}_{s>t},\\
X^{n+1}(s) &= y_t(s)\mathbb{1}_{s\le t} + \Bigg[y_t(t) + \int_0^{s-t} b(t+u, X^n_{(t+u)-})du\\
&\quad + \int_0^{s-t}\sigma(t+u, X^n_{(t+u)-})dW(u) + \int_0^{s-t}\int_{|z|<c}\bar{\gamma}(t+u, X^n_{(t+u)-}, z)\bar{\mu}(du,dz)\Bigg]\mathbb{1}_{s>t}.
\end{aligned}$$

By induction we will prove the existence of a constant $C = C(T, K_2)$ such that

$$\mathbb{E}\left[\sup_{u\in[0,s]} |X^n(u)|^2\right] \le C\left(1 + \sup_{u\in[0,t]} |y_t(u)|^2\right)e^{C(s-t)}$$

for all $T \ge t$, $n \ge 0$, $s \in [t,T]$. For $n = 0$:

$$\mathbb{E}\left[\sup_{u\in[0,s]} \left|X^0(u)\right|^2\right] = \mathbb{E}\left[\sup_{u\in[0,s]} |y_t(u)\mathbb{1}_{u\le t} + y_t(t)\mathbb{1}_{u>t}|^2\right] = \sup_{u\in[0,t]} |y_t(u)|^2 \le C\left(1 + \sup_{u\in[0,t]} |y_t(u)|^2\right)e^{C(s-t)},$$

for some constant $C > 1$. Now assume that the statement is true for some $n \in \mathbb{N}$. For $u \in [0, s-t]$, $s \in [t,T]$, we define

$$\begin{aligned}
I_1(u) &= \int_0^u b(t+v, X^n_{(t+v)-})dv,\\
I_2(u) &= \int_0^u \sigma(t+v, X^n_{(t+v)-})dW(v),\\
I_3(u) &= \int_0^u \int_{|z|<c} \bar{\gamma}(t+v, X^n_{(t+v)-}, z)\bar{\mu}(dv,dz).
\end{aligned}$$

Then, by using Cauchy-Schwarz inequality and $(C2)$ we obtain

$$\begin{aligned}
|I_1(u)|^2 &= \left|\int_0^u b(t+v, X^n_{(t+v)-})dv\right|^2 \le u\cdot\int_0^u \left|b(t+v, X^n_{(t+v)-})\right|^2 dv\\
&\le u \sup_{\tau\in[0,T]} K_2(\tau)\cdot\int_0^u \left(1 + \sup_{\tau\in[0,t]} |y_t(\tau)|^2 + \sup_{\tau\in[0,v]}\left|X^n_{(t+v)-}(t+\tau)\right|^2\right)dv.
\end{aligned}$$

Hence, setting $K_2 := \sup_{\tau\in[0,T]} K_2(\tau)$, leads to

$$\mathbb{E}\left[\sup_{u\in[0,s-t]} |I_1(u)|^2\right] \le T^2K_2\left(1 + \sup_{\tau\in[0,t]} |y_t(\tau)|^2\right) + TK_2\left(1 + \sup_{\tau\in[0,t]} |y_t(\tau)|^2\right)e^{C(s-t)}.$$

For I_2 we get with (C2)

$$\langle I_2\rangle(u) \le K_2\int_0^u \left(1 + \sup_{\tau\in[0,t]} |y_t(\tau)|^2 + \sup_{\tau\in[0,v]}\left|X^n_{(t+v)-}(t+\tau)\right|^2\right)dv,$$

for $u \in [0, s-t]$. By the Burkholder-Davis-Gundy inequality, there exist a constant Λ such that

$$\mathbb{E}\left[\sup_{u\in[0,s-t]} |I_2(u)|^2\right] \le \Lambda K_2 T\left(1 + \sup_{\tau\in[0,t]} |y_t(\tau)|^2\right) + \Lambda K_2\left(1 + \sup_{\tau\in[0,t]} |y_t(\tau)|^2\right)e^{C(s-t)}.$$

Note that similar bounds for the integrals I_1 and I_2 have been obtained in the proof of Theorem B.1 in Fournié (2010). Furthermore, with Doob's martingale inequality and Theorem 2.3.8 in Applebaum (2005), we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{u\in[0,s-t]}|I_3(u)|^2\right] &\leq 4K_2\mathbb{E}\left[\int_0^{s-t}\left(1+\sup_{\tau\in[0,t+v]}|X^n_{(t+v)-}(\tau)|^2\right)dv\right]\\ &\leq 4K_2T\left(1+\sup_{\tau\in[0,t]}|y_t(\tau)|^2\right)+4K_2\left(1+\sup_{\tau\in[0,t]}|y_t(\tau)|^2\right)e^{C(s-t)}.\end{aligned}$$

Together, this leads to

$$\begin{aligned}\mathbb{E}\left[\sup_{u\in[t,s]}\left|X^{n+1}(u)\right|^2\right] \leq\ & 20\cdot\max\{1,TK_2\}(1+T+\Lambda)\left(1+\sup_{\tau\in[0,t]}|y_t(\tau)|^2\right)e^{C(s-t)}\\ &+20K_2\cdot(1+T+\Lambda)\left(1+\sup_{\tau\in[0,t]}|y_t(\tau)|^2\right)e^{C(s-t)}.\end{aligned}$$

For

$$C=20\cdot(\max\{1,TK_2\}+K_2)(1+T+\Lambda),$$

this yields for all $n\in\mathbb{N}$, $s\in[t,T]$,

$$\mathbb{E}\left[\sup_{u\in[0,s]}|X^n(u)|^2\right]\leq C\left(1+\sup_{u\in[0,t]}|y_t(u)|^2\right)e^{C(s-t)}.$$

Now, define for $u\in[0,s-t]$ the processes

$$\begin{aligned}\bar{I}_1(u)&=\int_0^u\left[b(t+v,X^n_{(t+v)-})-b(t+v,X^{n-1}_{(t+v)-})\right]dv,\\ \bar{I}_2(u)&=\int_0^u\left[\sigma(t+v,X^n_{(t+v)-})-\sigma(t+v,X^{n-1}_{(t+v)-})\right]dW(v),\\ \bar{I}_3(u)&=\int_0^u\int_{|z|<c}\left[\bar{\gamma}(t+v,X^n_{(t+v)-},z)-\bar{\gamma}(t+v,X^{n-1}_{(t+v)-},z)\right]\bar{\mu}(dv,dz).\end{aligned}$$

Using the Cauchy-Schwarz inequality and the Lipschitz condition (C1) we obtain

$$\mathbb{E}\left[\sup_{u\in[0,s-t]}|\bar{I}_1(u)|^2\right]\leq T\sup_{\tau\in[0,T]}K_1(\tau)\cdot\int_0^{s-t}\mathbb{E}\left[\sup_{\tau\in[0,t+v]}\left|X^n_{(t+v)-}(\tau)-X^{n-1}_{(t+v)-}(\tau)\right|^2\right]dv.$$

By the Burkholder-Davis-Gundy inequality and (C1) it follows

$$\mathbb{E}\left[\sup_{u\in[0,s-t]}|\bar{I}_2(u)|^2\right]\leq \Lambda\sup_{\tau\in[0,T]}K_1(\tau)\cdot\int_0^{s-t}\mathbb{E}\left[\sup_{\tau\in[0,t+v]}\left|X^n_{(t+v)-}(\tau)-X^{n-1}_{(t+v)-}(\tau)\right|^2\right]dv.$$

Furthermore,

$$\begin{aligned}\mathbb{E}\left[\sup_{u\in[0,s-t]}|\bar{I}_3(u)|^2\right]&\leq 4\mathbb{E}\left[\int_0^{s-t}\int_{|z|<c}\left|\bar{\gamma}(t+v,X^n_{(t+v)-},z)-\bar{\gamma}(t+v,X^{n-1}_{(t+v)-},z)\right|^2\nu(dz)\,dv\right]\\ &\leq 4\mathbb{E}\left[\int_0^{s-t}K_1(t+v)\sup_{\tau\in[0,t+v]}|X^n_{(t+v)-}(\tau)-X^{n-1}_{(t+v)-}(\tau)|^2dv\right]\\ &\leq 4\sup_{\tau\in[0,T]}K_1(\tau)\cdot\int_0^{s-t}\mathbb{E}\left[\sup_{\tau\in[0,t+v]}|X^n_{(t+v)-}(\tau)-X^{n-1}_{(t+v)-}(\tau)|^2\right]dv.\end{aligned}$$

Together, this leads to

$$\mathbb{E}\left[\sup_{u\in[0,s-t]}\left|X^{n+1}(u)-X^{n}(u)\right|^{2}\right]\leq 3\left(T+\Lambda+4\right)\sup_{\tau\in[0,T]}K_{1}(\tau)\int_{0}^{s-t}\mathbb{E}\left[\sup_{\tau\in[0,t+v]}|X^{n}_{(t+v)-}(\tau)-X^{n-1}_{(t+v)-}(\tau)|^{2}\right]dv,$$

for all $s\in[t,T]$. Reiterating ensures that for all $s\in[t,T]$ and $K_1:=\sup_{\tau\in[0,T]}K_1(\tau)$

$$\mathbb{E}\left[\sup_{u\in[0,s-t]}\left|X^{n+1}(u)-X^{n}(u)\right|^{2}\right]\leq C^{*}\frac{\left[3\left(T+\Lambda+4\right)K_{1}\right]^{n}t^{n}}{n!}$$

with

$$C^{*}=\mathbb{E}\left[\sup_{u\in[0,s-t]}\left|X^{1}(u)-X^{0}(u)\right|^{2}\right]<\infty.$$

The Chebychev inequality provides

$$\mathbb{P}\left(\sup_{u\in[0,s-t]}\left|X^{n+1}(u)-X^{n}(u)\right|\geq\frac{1}{2^{n+1}}\right)\leq 4C^{*}\frac{\left[12\left(T+\Lambda+4\right)K_{1}T\right]^{n}}{n!}.$$

By applying the Borel-Cantelli lemma, we get

$$\mathbb{P}\left(\limsup_{n\to\infty}\sup_{u\in[0,s]}\left|X^{n+1}(u)-X^{n}(u)\right|\geq\frac{1}{2^{n+1}}\right)=0.$$

We have to show that $(X^n(s),n\in\mathbb{N})$ is convergent in L^2 for all $s\geq 0$. By the definition of X^n we know that X^n is adapted and càdlàg. Indeed we obtain for each $m,n\in\mathbb{N}$

$$\left\|X^{n}(u)-X^{m}(u)\right\|_{2}\leq\sum_{i=m+1}^{n}\left(C^{*}\frac{\left[3T\left(T+\Lambda+4\right)K_{1}\right]^{i}}{i!}\right)^{\frac{1}{2}}.$$

Thus $(X^n(s),n\in\mathbb{N})$ is a Cauchy sequence and hence convergent to some $X(s)\in L^2(\Omega,\mathscr{F},\mathbb{P})$, since the series on the right hand converges. Furthermore, we get

$$\left\|X(u)-X^{n}(u)\right\|_{2}\leq\sum_{i=n+1}^{\infty}\left(C^{*}\frac{\left[3T\left(T+\Lambda+4\right)K_{1}\right]^{i}}{i!}\right)^{\frac{1}{2}}$$

for $u\in[0,s-t]$ and $n\in\mathbb{N}_0$. So we receive that $(X^n,n\in\mathbb{N}_0)$ is almost surely uniformly convergent on $[0,s-t]$ to X, X is adapted and càdlàg. Now define a stochastic process

$$\begin{aligned}\tilde{X}(s)=y_t(s)\mathbb{1}_{s\leq t}+\Bigg[y_t(t)+\int_0^{s-t}b\left(t+u,X_{(t+u)-}\right)du\\+\int_0^{s-t}\sigma\left(t+u,X_{(t+u)-}\right)dW(s)+\int_0^{s-t}\int_{|z|<c}\bar{\gamma}\left(t+u,X_{(t+u)-},z\right)\bar{\mu}(du,dz)\Bigg]\mathbb{1}_{s>t}.\end{aligned}$$

It follows for $s>t$:

$$\begin{aligned}\tilde{X}(s)-X^n(s)=\int_0^{s-t}\left[b(t+u,X_{(t+u)-})-b(t+u,X^{n-1}_{(t+u)-})\right]du+\int_0^{s-t}\left[\sigma(t+u,X_{(t+u)-})-\sigma(t+u,X^{n-1}_{(t+u)-})\right]dW(s)\\+\int_0^{s-t}\int_{|z|<c}\left[\bar{\gamma}(t+u,X_{(t+u)-},z)-\bar{\gamma}(t+u,X^{n-1}_{(t+u)-},z)\right]\bar{\mu}(du,dz).\end{aligned}$$

We obtain

$$\mathbb{E}\left[\left|\tilde{X}(s)-X^{n}(s)\right|^{2}\right] \leq 3\tilde{C}K_{1}\int_{0}^{s-t}\mathbb{E}\left[\sup_{\tau\in[0,t+u]}\left|X_{(t+u)-}(\tau)-X^{n-1}_{(t+u)-}(\tau)\right|^{2}\right]du$$
$$\leq 3\tilde{C}K_{1}T\cdot\left(\sum_{i=n+1}^{n}\left(C^{*}\frac{\left[3T\left(T+\Lambda+4\right)K_{1}\right]^{i}}{i!}\right)^{\frac{1}{2}}\right)^{2}\to 0 \quad \text{for } n\to\infty.$$

Hence, for $\sup_{s\in[0,t]}|y_t(s)|^2<\infty$, we have a unique solution. □

Corollary 3.2. *Under (C1)-(C3) there exists a unique solution to (3.1).*

The proof is analogous to the proof of Theorem 6.2.9 in Applebaum (2005) for a non-path-dependent case. It uses the technique of interlacing, see Section 2.5 in Applebaum (2005).

We can rewrite (3.1) as

$$dX(t+s)=\tilde{b}\left(t+s,X_{(t+s)-}\right)ds+\sigma\left(t+s,X_{(t+s)-}\right)dW(s)$$
$$+\int_{\mathbb{R}^d}\tilde{\gamma}\left(t+s,X_{(t+s)-},z\right)\bar{\mu}(ds,dz),$$

with $\tilde{b}(t,x)=b(t,x)+\int_{|z|\geq c}\gamma(t,x,z)\nu(dz)$ and $\tilde{\gamma}(t,x,z)=\bar{\gamma}(t,x,z)\mathbb{1}_{\{|z|<c\}}+\gamma(t,x,z)\mathbb{1}_{\{|z|\geq c\}}$. Similar to the proof of Theorem 3.1 we get the following corollary.

Corollary 3.3. *If we modify condition (C2) by the existence of $K_2(s)>0$, such that for all $x\in D([0,T],\mathbb{R}^d)$ and each $s>t$*

$$|\tilde{b}(s,x)|^2+\|a(s,x,x)\|+\int_{\mathbb{R}^d}|\tilde{\gamma}(s,x,z)|^2\nu(dz)\leq K_2(s)(1+\sup_{u\leq s}|x(u)|^2) \tag{3.4}$$

we get

$$\mathbb{E}\left[\sup_{u\in[0,s]}|X(u)|^2\right]\leq C\left(1+\sup_{u\in[0,t]}|y_t(u)|^2\right)e^{C(s-t)}$$

for all $s\in[t,T]$.

We now move on to the proof of the functional Feynman-Kac theorem for jump diffusions with path-dependent coefficients. With an arbitrary but fixed $T>0$ and an appropriate constant $L>0$ we consider functions $f: D([0,T],\mathbb{R}^d)\to\mathbb{R}$, $g:[0,T]\times D([0,T],\mathbb{R}^d)\to\mathbb{R}$ and $k:[0,T]\times D([0,T],\mathbb{R}^d)\to[0,\infty)$ which are continuous with respect to the metric $\tilde{d}$ and satisfy

$$(i)\ |f(y)|\leq L(1+\sup_{u\in[0,T]}|y(u)|^2) \quad\text{or}\quad (ii)\ f(y)\geq 0 \quad \forall y\in D([0,T],\mathbb{R}^d) \tag{3.5}$$

as well as

$$(i)\ |g(s,y)|\leq L(1+\sup_{u\in[0,T]}|y(u)|^2) \quad\text{or}\quad (ii)\ g(s,y)\geq 0;\quad \forall 0\leq s\leq T,\ y\in D([0,T],\mathbb{R}^d). \tag{3.6}$$

Furthermore we consider the operator $\mathscr{A}_t$

$$(\mathscr{A}_t f)(y):=\frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d}a_{ik}(t,y,y)D^2_{ik}f(y;[t,T])+\sum_{i=1}^{d}\tilde{b}_i(t,y)D^1_i f(y;[t,T])$$
$$\int_{\mathbb{R}^d}\left[f(y^{\tilde{\gamma}(t,y,z)})-f(y)-\sum_{i=1}^{d}\tilde{\gamma}_i(t,y,z)D^1_i f(y;[t,T])\right]\nu(dz);\quad f\in C^2(D([0,T],\mathbb{R}^d)). \tag{3.7}$$

If f is a function of $t\in[0,\infty)$ and $y\in D([0,T],\mathbb{R}^d)$, then $(\mathscr{A}_t f)(t,y)$ is obtained by applying $\mathscr{A}_t$ to $f(t,\cdot)$. Denote by $\mathbb{E}^{t,y_t}[\cdot]$ the conditional expectation $\mathbb{E}^{t,y_t}[\cdot]=\mathbb{E}[\cdot|X_t=y]$ and by $\mathbb{P}^{t,y_t}(\cdot)$ the conditional probability $\mathbb{P}^{t,y_t}(A)=\mathbb{E}^{t,y_t}[1_A]$ for $t\geq 0$ and $y\in D([0,T],\mathbb{R}^d)$. In the proof of the following theorem we combine some of the ideas of the proofs of Theorem 7.6 in Karatzas and Shreve (1991) with the functional Itô formula from Levental et al. (2013) (see also Cont and Fournié (2010)) and our assumptions on the path-dependent coefficients of the underlying stochastic differential equation (3.1).

Remark 3.4. *Connection to other path processes. In the context of so called superprocesses, path valued processes were also considered in the ninties in Dynkin (1994), Dawson and Perkins (1991); Perkins (1993, 1995) . In the last two references, the generator of the path-valued processes in terms of martingale problems is described. This means that elements in the domain of our $\mathscr{A}_t$ are described. In equation (1.1) of Perkins (1995) the function $f(y)$ is of the form $\psi(y(t_1,..t_n))$, $f(y) = \psi(y(t_1),..y(t_n))$ $\psi \in C_b^2(\mathbb{R}^{nd})$ and the differentials above can then easily be expressed as the usual derivatives of ψ. It was also shown in Perkins (1993, 1995) that the corresponding martingale problem is well-posed.*

Theorem 3.5. *Under the preceding assumptions and the requirements of Corollary 3.3, suppose that $v : [0,T] \times D([0,T],\mathbb{R}^d) \to \mathbb{R}$ satisfies the assumptions of Theorem 2.2 and the Cauchy problem*

$$-D_t v + kv = \mathscr{A}_t v + g; \quad in\ [0,T) \times D([0,T],\mathbb{R}^d), \tag{3.8}$$

$$v(T,y) = f(y); \quad y \in D([0,T],\mathbb{R}^d), \tag{3.9}$$

as well as the polynomial growth condition

$$\sup_{0\le t\le T} |v(t,y)| \le M(1 + \sup_{0\le t\le T} |y(t)|^2); \quad y \in D([0,T],\mathbb{R}^d), \tag{3.10}$$

for some $M > 0$. Then $v(t,y)$ admits the stochastic representation

$$v(t,y_t) = \mathbb{E}^{t,y_t}\left[f(X_T)\exp\left\{-\int_t^T k(u,X_u)du\right\} + \int_t^T g(s,X_s)\exp\left\{-\int_t^s k(u,X_u)du\right\}ds\right] \tag{3.11}$$

on $[0,T] \times D([0,T],\mathbb{R}^d)$; in particular, such a solution is unique.

Proof. We apply the functional Itô formula from Theorem 2.2, respectively Remark 2.3, to $v(s,X_s)\exp\{-\int_t^s k(u,X_u)du\}$, $s \in [t,T]$ and obtain

$$\begin{aligned}
d\left[v(s,X_s)\exp\left\{-\int_t^s k(u,X_u)du\right\}\right] &= -v(s,X_s)k(s,X_s)\exp\left\{-\int_t^s k(u,X_u)du\right\}ds \\
&+ \exp\left\{-\int_t^s k(u,X_u)du\right\}\left[D_s v(s,X_{s-};[s,T])ds + \sum_{i=1}^d D_i^1 v(s,X_{s-};[s,T])dX^i(s)\right. \\
&\left. + \frac{1}{2}\sum_{i,j} D_{ij}^2 v(s,X_{s-};[s,T])d\left\langle X^{i,c}(s),X^{j,c}(s)\right\rangle + \left(v(s,X_s) - v(s,X_{s-}) - \sum_{i=1}^d D_i^1 v(s,X_{s-};[s,T])\Delta X^i(s)\right)\right],
\end{aligned}$$

where X^c is the continuous part of the semimartingale X and $\Delta X(t) = X(t) - X(t-)$. Plugging in X from (3.1) leads to

$$\begin{aligned}
&= \exp\left\{-\int_t^s k(u,X_u)du\right\}\left[-v(s,X_s)k(s,X_s)ds + D_s v(s,X_{s-};[s,T])ds\right. \\
&+ \sum_{i=1}^d D_i^1 v(s,X_{s-};[s,T])b_i(s,X_{s-})ds + \sum_{i=1}^d D_i^1 v(s,X_{s-};[s,T])\sigma_i(s,X_{s-})dW^i(s) \\
&+ \frac{1}{2}\sum_{i,j} D_{ij}^2 v(s,X_{s-};[s,T])a_{ij}(s,X_{s-},X_{s-})ds + \sum_{i=1}^d \int_{|z|\ge c} D_i^1 v(s,X_{s-};[s,T])\gamma_i(s,X_{s-},z)\mu(ds,dz) \\
&\left. + \sum_{i=1}^d \int_{|z|<c} D_i^1 v(s,X_{s-};[s,T])\bar{\gamma}_i(s,X_{s-},z)\bar{\mu}(ds,dz) + (v(s,X_s) - v(s,X_{s-}) - \sum_{i=1}^d D_i^1 v(s,X_{s-};[s,T])\Delta X^i(s))\right]
\end{aligned}$$

After sorting the terms and using the representation of the jumps by the jump measure μ we obtain

$$
\begin{aligned}
&= \exp\left\{-\int_t^s k(u,X_u)du\right\}\Bigg[\Bigg(-v(s,X_s)k(s,X_s)+D_s v(s,X_{s-};[s,T])\\
&\quad+\sum_{i=1}^d D_i^1 v(s,X_{s-};[s,T])\left[b_i(s,X_{s-})+\int_{|z|\ge c}\gamma_i(s,X_{s-}z)\nu(dz)\right]+\frac{1}{2}\sum_{i,j}D_{ij}^2 v(s,X_{s-};[s,T])a_{ij}(s,X_{s-},X_{s-})\\
&\quad+\int_{|z|\ge c}\left[v(s,X_{s-}^{\gamma(s,X_{s-},z)})-v(s,X_{s-})-\sum_{i=1}^d D_i^1 v(s,X_{s-};[s,T])\gamma_i(s,X_{s-},z)\right]\nu(dz)\\
&\quad+\int_{|z|<c}\left[v(s,X_{s-}^{\bar\gamma(s,X_{s-},z)})-v(s,X_{s-})-\sum_{i=1}^d D_i^1 v(s,X_{s-};[s,T])\bar\gamma_i(s,X_{s-},z)\right]\nu(dz)\Bigg)ds\\
&\quad+\sum_{i=1}^d D_i^1 v(s,X_{s-};[s,T])\sigma_i(s,X_{s-})dW^i(s)+\int_{|z|\ge c}[v(s,X_{s-}^{\gamma(s,X_{s-},z)})-v(s,X_{s-})]\bar\mu(ds,dz)\\
&\quad+\int_{|z|<c}[v(s,X_{s-}^{\bar\gamma(s,X_{s-},z)})-v(s,X_{s-})]\bar\mu(ds,dz).
\end{aligned}
$$

Together with (3.8) this implies

$$
\begin{aligned}
&d\left[v(s,X_s)\exp\left\{-\int_t^s k(u,X_u)du\right\}\right]\\
&= \exp\left\{-\int_t^s k(u,X_u)du\right\}\Bigg[-g(s,X_{s-})ds+\sum_{i=1}^d D_i^1 v(s,X_{s-};[s,T])\sigma_i(s,X_{s-})dW^i(s)\\
&\quad+\int_{\mathbb{R}^d}[v(s,X_{s-}^{\tilde\gamma(s,X_{s-},z)})-v(s,X_{s-})]\bar\mu(ds,dz).
\end{aligned}
$$

We define $\tau_n=\inf\{s\ge t; \|X(s)\|\ge n\}$ and integrate the above expression over $[t,T\wedge\tau_n]$. This leads to

$$
\begin{aligned}
&v(T\wedge\tau_n,X_{T\wedge\tau_n})\exp\left\{-\int_t^{T\wedge\tau_n}k(u,X_u)du\right\}-v(t,X_t)=-\int_t^{T\wedge\tau_n}g(s,X_{s-})\exp\left\{-\int_t^s k(u,X_{u-})du\right\}ds\\
&+\int_t^{T\wedge\tau_n}\exp\left\{-\int_t^s k(u,X_{u-})du\right\}\sum_{i=1}^d\sigma_i(s,X_{s-})D_i^1 v(s,X_{s-};[s,T])dW^i(s)\\
&+\int_t^{T\wedge\tau_n}\int_{|z|\ge c}[v(s,X_{s-}^{\gamma(s,X_{s-},z)})-v(s,X_{s-})]\bar\mu(ds,dz)+\int_t^{T\wedge\tau_n}\int_{|z|<c}[v(s,X_{s-}^{\bar\gamma(s,X_{s-},z)})-v(s,X_{s-})]\bar\mu(ds,dz).
\end{aligned}
$$

The stochastic integral and the jump part integrals have expectation zero. Thus taking expectations above leads to

$$
\begin{aligned}
v(t,X_t)=v(t,y_t)=\mathbb{E}^{t,y_t}&\left[v(\tau_n,X_{\tau_n})\exp\left\{-\int_t^{\tau_n}k(u,X_{u-})du\right\}1_{\{\tau_n\le T\}}\right]\\
&+\mathbb{E}^{t,y_t}\left[f(X_T)\exp\left\{-\int_t^T k(u,X_{u-})du\right\}1_{\{\tau_n>T\}}\right]\\
&+\mathbb{E}^{t,y_t}\left[\int_t^{T\wedge\tau_n}g(s,X_{s-})\exp\left\{-\int_t^s k(u,X_{u-})du\right\}ds\right].
\end{aligned}
\tag{3.12}
$$

Now, either by the dominated convergence theorem together with (3.6)(i) and (3.3) or by the monotone convergence theorem together with (3.6)(ii) the last term in (3.12) converges to

$$
\mathbb{E}^{t,y_t}\left[\int_t^T g(s,X_{s-})\exp\left\{-\int_t^s k(u,X_{u-})du\right\}ds\right].
$$

Similarly the second term in (3.12) converges to $\mathbb{E}^{t,y}[f(X_T)\exp\{-\int_t^T k(u,X_{u-})du\}]$ by the dominated convergence theorem together with (3.5)(i) and (3.3) or by the monotone convergence theorem together with (3.5)(ii). By condition

(3.10) and (3.3) the first term in (3.12) is bounded by

$$\mathbb{E}^{t,y_t}\left[|v(\tau_n,X_{\tau_n})|\mathbb{1}_{\{\tau_n\le T\}}\right]\le \mathbb{E}^{t,y_t}\left[\sup_{0\le t\le T}|v(t,X_t)|\mathbb{1}_{\{\tau_n\le T\}}\right]\le C(1+\sup_{u\in[0,t]}|y_t(u)|^2)e^{C(T-t)}\mathbb{P}^{t,y_t}(\tau_n\le T). \tag{3.13}$$

Together with (3.3) and the Chebychev inequality the probability $\mathbb{P}^{t,y_t}(\tau_n\le T)$ satisfies

$$\mathbb{P}^{t,y_t}(\tau_n\le T)=\mathbb{P}^{t,y_t}\left(\sup_{t\le\theta\le T}\|X_\theta\|\ge n\right)\le n^{-2}\mathbb{E}^{t,y_t}\left[\sup_{t\le\theta\le T}\|X_\theta\|^2\right]\le Cn^{-2}\left(1+\sup_{u\in[0,t]}|y_t(u)|^2\right)e^{C(T-t)}$$

As $n\to\infty$ we see that the right hand side of (3.13) converges to zero which implies that the first term in (3.12) converges to zero. Finally we obtain

$$v(t,y_t)=\mathbb{E}^{t,y_t}\left[\int_t^T g(s,X_{s-})\exp\left\{-\int_t^s k(u,X_{u-})du\right\}ds+f(X_T)\exp\left\{-\int_t^T k(u,X_{u-})du\right\}\right].$$

□

Remark 3.6. *Note that we did not use the classical Feynman-Kac theorem for the proof of the previous theorem. If we use it there is an alternative way for a proof by similar path approximation arguments as in Levental et al. (2013). We will outline the idea of this proof. Define* $t_k^n=\frac{k}{n}T$, $k=1,\dots,n$, $t_n(t)=\inf\{t_k^n: t_k^n>t\}$, $\alpha_0=0$ *and*

$$\alpha_{k+1}^n=\inf\left\{t>\alpha_k^n:|X(t)-X(\alpha_k^n)|>\frac{1}{n}\right\}\wedge t_n(\alpha_k)\wedge T.$$

Since X *is a càdlàg semimartingale we have* $\alpha_{k+1}^n>\alpha_k^n$ *a.s. on the event* $\{\alpha_k^n<T\}$*. Also there exists* $k<\infty$ *such that* $\alpha_k^n=T$*. Define the approximation of the path of* X *on* $[0,t]$ *by*

$$App^n(X_t)=\left[\sum_{l=0}^{k-1}X(\alpha_l^n)\mathbb{1}_{[\alpha_l^n,\alpha_{l+1}^n]}\right]+X(\alpha_k^n)\mathbb{1}_{[\alpha_k^n,t_n(t)]}+X(t)\mathbb{1}_{[t_n(t),T]}\quad if\quad t\in[\alpha_k^n,\alpha_k^{n+1}).$$

According to Levental et al. (2013), we have $F(App^n(X_t))=f(X(t),A)$ *if* $t\in[\alpha_k^n,\alpha_k^{n+1})$ *for some function* $f:\mathbb{R}^d\times\Omega\to\mathbb{R}$ *with* $A\in\mathscr{F}_{\alpha_k^n}$*. Then* $\nabla_x f(X(t-))=D^1F(App^n(X_{t-});[t_n(t),T])$ *and* $\nabla_{xx}f(X(t-)=D^2F(App^n(X_{t-});[t_n(t),T])$ *if* $t\in[\alpha_k^n,\alpha_k^{n+1})$*. Consider*

$$\begin{aligned}dX(t+s)&=b(t+s,App^n(X_{(t+s)-}))ds+\sigma(t+s,App^n(X_{(t+s)-}))dW(s)\\&\quad+\int_{|z|<c}\bar{\gamma}(t+s,App^n(X_{(t+s)-}),z)\bar{\mu}(ds,dz)+\int_{|z|\ge c}\gamma(t+s,App^n(X_{(t+s)-}),z)\mu(ds,dz),\end{aligned}\tag{3.14}$$

with initial value

$$App^n(X_t)=App^n(y_t).$$

Then, for the process $Z(t)=(t,X(t))$ *we have* $v(App^n(Z_t))=\bar{v}(t,X(t),A)$ *if* $t\in[\alpha_k^n,\alpha_k^{n+1})$ *for some function* $\bar{v}:[0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$*, where* $A\in\mathscr{F}_{\alpha_k^n}$*. Under the assumptions of Theorem 3.5 we know that* v *satisfies the Cauchy problem on* $[0,T]\times D([0,T],\mathbb{R}^d)$*. Thus* $\bar{v}(t,X(t),A)=v(App^n(Z_t))$ *satisfies*

$$-D_t\bar{v}+k\bar{v}=\mathscr{A}_t\bar{v}+g; \tag{3.15}$$

and the classical Feynman-Kac yields that the solution to (3.15) *admits the stochastic representation*

$$\begin{aligned}\bar{v}(t,X(t))=v(t,App^n(y_t))=\mathbb{E}^{t,App^n(y_t)}&\left[f(App^n(X_T))\exp\left\{-\int_t^T k(u,App^n(X_u)du\right\}\right.\\&\left.\int_t^T g(s,App^n(X_s)\exp\left\{-\int_t^s k(u,App^n(X_u)du\right\}ds\right]\end{aligned}$$

if $t \in [\alpha_k^n, \alpha_k^{n+1})$. Then it follows from the assumptions on f and g together with the dominated convergence theorem, respectively the monotone convergence theorem

$$\mathbb{E}^{t,App^n(y_t)}\left[f(App^n(X_T))\exp\left\{-\int_t^T k(u,App^n(X_u)du\right\}+\int_t^T g(s,App^n(X_s)\exp\left\{-\int_t^s k(u,App^n(X_u)du\right\}ds\right]$$
$$\overset{n\to\infty}{\longrightarrow}\mathbb{E}^{t,y_t}\left[f(X_T)\exp\left\{-\int_t^T k(u,X_u)du\right\}+\int_t^T g(s,X_s)\exp\left\{-\int_t^s k(u,X_u)du\right\}ds\right]$$

and since $v(t,App^n(y_t)) \to v(t,y_t)$ the statement of Theorem 3.5 follows.

4 Credit Value Adjustment PIDE

Bilateral counterparty risk gained visibility in the wake of the global financial crisis. It describes the risk that the issuer of a derivative contract or the counterparty may default prior to expiry and thus may fail to make future payments. The valuation of this type of risk, which is implicitly embedded in derivative contracts, is of great significance in current markets and Credit Value Adjustment (CVA) has become a very important research topic in mathematical finance, see Piterbarg (2010), Burgard and Kjaer (2011), Crépey (2012) and Henry-Labordère (2012). In Burgard and Kjaer (2011) the authors establish a framework for calculating CVAs in a Markovian setup in terms of a bilateral jump-to-default model. It is a framework for the valuation of the bilateral counterparty risk of a European derivative which includes funding considerations into the financing of the hedge positions. The authors use replication techniques (hedging arguments) and Itô's formula to derive extensions of the Black-Scholes partial differential equation (PDE) in conjunction with bilateral counterparty risk. By using the classical Feynman-Kac representation they decompose the risky derivative value into a counterparty risk free part and a CVA part. Other literature studies the CVA problem in conjunction with FBSDEs or with marked branching diffusions, see for instance Crépey (2012) and Henry-Labordère (2012). However, all the previously mentioned approaches to CVA are connected by the semi-linear CVA-PDE and they all have in common that they are designed for the valuation of counterparty risk of derivatives with a payoff at maturity T which depends only on the value of the underlying asset at time T. Thus they all exclude path-dependent derivatives like Asian options, Lookback options or Russian options. Our approach will overcome this problem in a non-Markovian setting.

Burgard and Kjaer (2011) have considered the market model

$$\begin{aligned} dP^R(t) &= r(t)P^R(t)dt, \\ dP^B(t) &= r^B(t)P^B(t)dt - P^B(t)dJ^B(t), \\ dP^C(t) &= r^C(t)P^C(t)dt - P^C(t)dJ^C(t), \\ dS(t) &= b(t)S(t)dt + \bar{\sigma}(t)S(t)dW(t), \end{aligned}$$

where $r(t) > 0$, $r^B(t) > 0$, $r^C(t) > 0$, $\bar{\sigma}(t) > 0$ are deterministic functions of t and J^B and J^C are two independent point processes that jump from 0 to 1 on default of party B respectively C. In this model P^R represents a default risk-free zero coupon bond, P^B and P^C are default risky, zero recovery, zero-coupon bonds of party B, respectively C and S represents the asset with no default risk. In this context party B is referred to as the seller and C as the counterparty. In contrast to this approach our results from the previous section enable us to study a more general market model which allows for path-dependent coefficients of the underlying SDEs. Furthermore we are able to add a jump component to the SDE of S. We will study the market model

$$dP^R(t) = r(t,P^R_{t-})dt, \tag{4.16}$$
$$dP^B(t) = r^B(t,P^B_{t-})dt - p^B(t,P^B_{t-})dJ^B(t), \tag{4.17}$$
$$dP^C(t) = r^C(t,P^C_{t-})dt - p^C(t,P^C_{t-})dJ^C(t), \tag{4.18}$$
$$dS(t) = \mu(t,S_{t-})dt + \sigma(t,S_{t-})dW(t) + \int_{|z|<c}\bar{\gamma}(t,S_{t-},z)\bar{\mu}(dt,dz) + \int_{|z|\geq c}\gamma(t,S_{t-},z)\mu(dt,dz), \tag{4.19}$$

and derive, with the help of the functional Itô formula and the functional Feynman-Kac theorem from the previous section, the corresponding Credit Value Adjustment PIDE. Our approach is general enough to study path-dependent derivatives, like the Asian option or the Russian option, in conjunction with CVA, which was not possible in Burgard and Kjaer (2011). To ensure existence and uniqueness of solutions to (4.16)-(4.19) we will assume throughout this section that the SDE coefficients satisfy the properties (C1)-(C3). Furthermore, we assume that $r(t,x) > 0$, $r^B(t,x) > 0$, $r^C(t,x) > 0$, $p^B(t,x) > 0$ and $p^C(t,x) > 0$ for all $(t,x) \in [0,T] \times D([0,T],\mathbb{R})$. Moreover, let the three jump processes be independent such that simultaneous jumps have probability zero. As in Burgard and Kjaer (2011) we assume that B and C enter a derivative on asset S. But in our approach the amount that the derivative pays seller B at maturity T may depend on the whole path of asset S on $[0,T]$. We denote this amount by $H(S_T)$ where $H(S_T) \geq 0$ means that seller B receives cash or an asset from C. We denote by $\hat{V}(t,X_t)$ the value of this derivative to the seller B at time t and simplify our notation by $X = (S,P^B,P^C)$ with $D^1_S\hat{V}(t,X_t;[t,T]) := D^1_1\hat{V}(t,X_t;[t,T])$. By $V(t,X_t)$ we denote the value to the seller B of the same derivative if it were a transaction between two default free parties. We adapt the boundary conditions from Burgard and Kjaer (2011), which in our approach are path-dependent and given by

$$\hat{V}^B(t,X_t) = \hat{V}(t,S_t,(P^B_t)^{-p^B(t,P^B_{t-})},P^C_t) = M^+(t,S_t) + R^B(t)M^-(t,S_t),$$

if seller B defaults first and

$$\hat{V}^C(t,X_t) = \hat{V}(t,S_t,P^B_t,(P^C_t)^{-p^C(t,P^C_{t-})}) = R^C(t)M^+(t,S_t) + M^-(t,S_t)$$

if the counterparty C defaults first.

$$R^B(t) = \frac{(P^B_t)^{-p^B(t,P^B_{t-})}(t)}{P^B_t(t)} \quad \text{and} \quad R^C(t) = \frac{(P^C_t(t))^{-p^C(t,P^C_{t-})}(t)}{P^C_t(t)}$$

are recoveries (note that these are constant values in Burgard and Kjaer (2011)) and $M(t,S_t)$ stands for the mark-to-market value of the derivative to the seller at default, specified for instance by the ISDA Master Agreement mentioned in Burgard and Kjaer (2011). In contrast to the complete market model of Burgard and Kjaer (2011) we have an incomplete market model because we allow S to have jumps. We know from Merton (1976), see also Chapter 10 in Cont and Tankov (2004), that we cannot hedge the risk of the jumps of S and thus we construct a hedging portfolio Π such that we can at least hedge all other risks. This means we construct Π as

$$\begin{aligned}
&\hat{V}(t,X_t) + \Pi(t,X_t) \\
&= \int_0^t \int_{\mathbb{R}} \left[\hat{V}(t,S_{t-}^{\tilde{\gamma}(t,S_{t-},z)},P^B_{t-},P^C_{t-}) - \hat{V}(t,X_{t-}) - D^1_S\hat{V}(t,X_{t-};[t,T])\tilde{\gamma}(t,S_{t-},z)\right] \bar{\mu}(dt,dz),
\end{aligned}$$

where the self-financing portfolio Π is given by

$$d\Pi(t,X_{t-}) = \delta(t,X_{t-})dS(t) + \alpha^B(t,X_{t-})dP^B(t) + \alpha^C(t,X_{t-})dP^C(t) + d\beta(t). \tag{4.20}$$

$d\beta$ is of the form

$$d\beta(t) = \left[-r\left(t,P^R_{t-}\right)\hat{V}(t,X_{t-}) + u\left(\delta(t,X_{t-}),\alpha^B(t,X_{t-}),\alpha^C(t,X_{t-})\right) - \delta(t,X_{t-})\,\varepsilon(t,S_{t-})\right]dt,$$

where ε is the difference between the rate of the dividend income and the rate of financing costs. For a specification of the function u, see page 6 in Burgard and Kjaer (2011). This implies $\hat{V}(t,X_t) + \hat{\Pi}(t,X_t) = 0$, where

$$\begin{aligned}
&\hat{\Pi}(t,X_t) = \Pi(t,X_t) \\
&- \int_0^t \int_{\mathbb{R}} \left[\hat{V}(t,S_{t-}^{\tilde{\gamma}(t,S_{t-},z)},P^B_{t-},P^C_{t-}) - \hat{V}(t,X_{t-}) - D^1_S\hat{V}(t,X_{t-};[t,T])\tilde{\gamma}(t,S_{t-},z)\right] \bar{\mu}(dt,dz).
\end{aligned}$$

With (4.20), (4.17) and (4.18) it follows

$$
\begin{aligned}
&-d\hat{V}(t,X_{t-}) = d\hat{\Pi}(t,X_t)\\
&= d\Pi(t,X_t) - \int_{\mathbb{R}} \left[\hat{V}(t,S_{t-}^{\tilde{\gamma}(t,S_{t-},z)},P_{t-}^B,P_{t-}^C) - \hat{V}(t,X_{t-}) - D_S^1\hat{V}(t,X_{t-};[t,T])\tilde{\gamma}(t,S_{t-},z)\right]\bar{\mu}(dt,dz)\\
&= \delta(t,X_{t-})dS(t) + \alpha^B(t,X_{t-})dP^B(t) + \alpha^C(t,X_{t-})dP^C(t) + d\beta(t)\\
&\quad - \int_{\mathbb{R}} \left[\hat{V}(t,S_{t-}^{\tilde{\gamma}(t,S_{t-},z)},P_{t-}^B,P_{t-}^C) - \hat{V}(t,X_{t-}) - D_S^1\hat{V}(t,X_{t-};[t,T])\tilde{\gamma}(t,S_{t-},z)\right]\bar{\mu}(dt,dz)\\
&= \delta(t,X_{t-})\mu(t,S_{t-})dt + \delta(t,X_{t-})\sigma(t,S_{t-})dW(t) + \int_{|z|<c}\delta(t,X_{t-})\tilde{\gamma}(t,S_{t-},z)\bar{\mu}(dt,dz)\\
&\quad + \int_{|z|\geq c}\delta(t,X_{t-})\gamma(t,S_{t-},z)\mu(dt,dz)\\
&\quad + \alpha^B(t,X_{t-})r^B(t,P_{t-}^B)dt - \alpha^B(t,X_{t-})p^B(t,P_{t-}^B)dJ^B(t) + \alpha^C(t,X_{t-})r^C(t,P_{t-}^C)dt - \alpha^C(t,X_{t-})p^C(t,P_{t-}^C)dJ^C(t)\\
&\quad - r(t,P_{t-}^R)\hat{V}(t,X_{t-})dt + u(\delta(t,X_{t-}),\alpha^B(t,X_{t-}),\alpha^C(t,X_{t-}))dt - \delta(t,X_{t-})\varepsilon(t,S_{t-})dt\\
&\quad - \int_{\mathbb{R}} \left[\hat{V}(t,S_{t-}^{\tilde{\gamma}(t,S_{t-},z)},P_{t-}^B,P_{t-}^C) - \hat{V}(t,X_{t-}) - D_S^1\hat{V}(t,X_{t-};[t,T])\tilde{\gamma}(t,S_{t-},z)\right]\bar{\mu}(dt,dz).
\end{aligned}
\tag{4.21}
$$

Heuristically we assume that Theorem 2.2 can be applied to the function $\hat{V}(t,X_t)$ and we get with

$$
\begin{aligned}
\Delta\hat{V}^B(t,X_{t-}) &:= \hat{V}^B(t,X_{t-}) - \hat{V}(t,X_{t-}),\\
\Delta\hat{V}^C(t,X_{t-}) &:= \hat{V}^C(t,X_{t-}) - \hat{V}(t,X_{t-}),
\end{aligned}
$$

$$
\begin{aligned}
&-d\hat{V}(t,X_{t-})\\
&= -D_t\hat{V}(t,X_{t-};[t,T])dt - D_S^1\hat{V}(t,X_{t-};[t,T])\mu(t,S_{t-})dt - D_S^1\hat{V}(t,X_{t-};[t,T])\sigma(t,S_{t-})dW(t)\\
&\quad - \int_{|z|<c} D_S^1\hat{V}(t,X_{t-};[t,T])\tilde{\gamma}(t,S_{t-},z)\bar{\mu}(dt,dz) - \int_{|z|\geq c} D_S^1\hat{V}(t,X_{t-};[t,T])\gamma(t,S_{t-},z)\mu(dt,dz)\\
&\quad - \int_{\mathbb{R}} \left[\hat{V}(t,S_{t-}^{\tilde{\gamma}(t,S_{t-},z)},P_{t-}^B,P_{t-}^C) - \hat{V}(t,X_{t-}) - D_S^1\hat{V}(t,X_{t-};[t,T])\tilde{\gamma}(t,S_{t-},z)\right]\bar{\mu}(dt,dz)\\
&\quad - \int_{\mathbb{R}} \left[\hat{V}(t,S_{t-}^{\tilde{\gamma}(t,S_{t-},z)},P_{t-}^B,P_{t-}^C) - \hat{V}(t,X_{t-}) - D_S^1\hat{V}(t,X_{t-};[t,T])\tilde{\gamma}(t,S_{t-},z)\right]\nu(dz)dt\\
&\quad - \frac{1}{2}D_{SS}^2\hat{V}(t,X_{t-};[t,T])\sigma^2(t,S_{t-})dt - \Delta\hat{V}^B(t,X_{t-})dJ^B(t) - \Delta\hat{V}^C(t,X_{t-})dJ^C(t).
\end{aligned}
\tag{4.22}
$$

Choosing α^B, α^C and δ as

$$
\begin{aligned}
\delta(t,X_{t-}) &= -D_S^1\hat{V}(t,X_{t-};[t,T]),\\
\alpha^B(t,X_{t-}) &= \frac{\Delta\hat{V}^B(t,X_{t-})}{p^B\left(t,P_{t-}^B\right)} = \frac{\hat{V}^B(t,X_{t-}) - \hat{V}(t,X_{t-})}{p^B\left(t,P_{t-}^B\right)},\\
\alpha^C(t,X_{t-}) &= \frac{\Delta\hat{V}^C(t,X_{t-})}{p^C\left(t,P_{t-}^C\right)} = \frac{\hat{V}^C(t,X_{t-}) - \hat{V}(t,X_{t-})}{p^C\left(t,P_{t-}^C\right)}
\end{aligned}
\tag{4.23}
$$

leads to the elimination of all remaining risks such that, if we define the generator $\mathscr{A}$ by

$$
\begin{aligned}
\mathscr{A}_t\hat{V}(t,X_{t-}) &= \left[\frac{1}{2}D_{SS}^2\hat{V}(t,X_{t-};[t,T])\sigma^2(t,S_{t-}) + D_S^1\hat{V}(t,X_{t-};[t,T])\varepsilon(t,S_{t-})\right]dt\\
&\quad + \int_{\mathbb{R}} \left[\hat{V}(t,S_{t-}^{\tilde{\gamma}(t,S_{t-},z)},P_{t-}^B,P_{t-}^C) - \hat{V}(t,X_{t-}) - D_S^1\hat{V}(t,X_{t-};[t,T])\tilde{\gamma}(t,S_{t-},z)\right]\nu(dz)dt
\end{aligned}
\tag{4.24}
$$

it follows that $\hat{V}$ satisfies

$$
\begin{aligned}
-D_t\hat{V}(t,X_{t-};[t,T]) + r\left(t,P_{t-}^R\right)\hat{V}(t,X_{t-}) &= \mathscr{A}_t\hat{V}(t,X_{t-}) + g(t,X_{t-})\\
\hat{V}(T,X_T) &= H(S_T),
\end{aligned}
\tag{4.25}
$$

with

$$g\left(t,X_{t-}\right)=u\left(-D_S^1\hat{V}\left(t,X_{t-};[t,T]\right),\frac{\hat{V}^B\left(t,X_{t-}\right)-\hat{V}\left(t,X_{t-}\right)}{p^B(t,P^B_{t-})},\frac{\hat{V}^C\left(t,X_{t-}\right)-\hat{V}\left(t,X_{t-}\right)}{p^C(t,P^C_{t-})}\right)$$
$$+\frac{\hat{V}^B\left(t,X_{t-}\right)-\hat{V}\left(t,X_{t-}\right)}{p^B(t,P^B_{t-})}r^B(t,P^B_{t-})+\frac{\hat{V}^C\left(t,X_{t-}\right)-\hat{V}\left(t,X_{t-}\right)}{p^C(t,P^C_{t-})}r^C(t,P^C_{t-}).$$

This implies α^B, α^C and δ define our hedging strategy which eliminates all risks except the jump risk of S and leads to the corresponding PIDE (4.25). The application of Theorem 3.5 implies that $\hat{V}$ has the representation

$$\hat{V}\left(t,X_t\right)=\mathbb{E}^{t,y_t}\left[H\left(S_T\right)\exp\left\{-\int_t^T r\left(u,P^R_{u-}\right)du\right\}+\int_t^T g\left(s,X_{s-}\right)\exp\left\{-\int_t^T r\left(u,P^R_{u-}\right)du\right\}ds\right]. \tag{4.26}$$

Similar to the argumentation in Burgard and Kjaer (2011) the default-risk-free value V satisfies

$$\begin{aligned}D_tV\left(t,X_{t-};[t,T]\right)+\mathscr{A}_tV\left(t,X_{t-}\right)-r(t,P^R_{t-})V\left(t,X_{t-}\right)&=0\\ V\left(T,X_T\right)&=H\left(S_T\right).\end{aligned} \tag{4.27}$$

If we now write $\hat{V}(t,X_t)=V(t,X_t)+U(t,X_t)$, where $U(t,X_t)$ denotes the Credit Value Adjustment (CVA), this leads us to the corresponding CVA-PIDE

$$\begin{aligned}-D_tU\left(t,X_{t-};[t,T]\right)+r(t,P^R_{t-})U\left(t,X_{t-}\right)&=\mathscr{A}_tU\left(t,X_{t-}\right)+g\left(t,X_{t-}\right)\\ U\left(T,X_T\right)&=0.\end{aligned} \tag{4.28}$$

Note that g depends on $\hat{V}$. With our functional Feynman-Kac formula we get for U the representation

$$U\left(t,X_t\right)=\mathbb{E}^{t,y_t}\left[\int_t^T g\left(s,X_{s-}\right)\exp\left\{-\int_t^T r\left(u,P^R_{u-}\right)du\right\}ds\right]. \tag{4.29}$$

If we remove the jumps of the asset price process S then the market model is complete and the same hedging strategy, given by δ, α^B and α^C in (4.23) eliminates all risks. The generator in (4.24) reduces to

$$\overline{\mathscr{A}_t}\hat{V}(t,X_{t-})=D_S^1\hat{V}(t,X_{t-};[t,T])\varepsilon(t,S_{t-})+\frac{1}{2}D^2_{SS}\hat{V}(t,X_{t-};[t,T])\sigma^2(t,S_{t-})$$

such that the PIDE in (4.25) changes to a PDE with $\overline{\mathscr{A}_t}$, but the representations in (4.26) and (4.29) remain the same. This directly generalizes the approach of Burgard and Kjaer (2011) and the pricing of path-dependent derivatives in a bilateral counterparty setting becomes possible.

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