Dynamic systemic risk measures for bounded discrete-time processes

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2016

Abstract

The question of measuring and managing systemic risk - especially in view of the recent financial crises - became more and more important. We study systemic risk by taking the perspective of a financial regulator and considering the axiomatic approach originally introduced in Chen et al. (2013) and extended in Kromer et al. (2014). The aim of this paper is to generalize the static approach in Kromer et al. (2014) and analyze systemic risk measures in a dynamic setting. We work in the framework of Cheridito et al. (2006) who consider risk measures for bounded discrete-time processes. Apart from the possibility to consider the "evolution of financial values", another important advantage of the dynamic approach is the possibility to incorporate information in the risk measurement and management process. In context of this dynamic setting we also discuss the arising question of time-consistency for our dynamic systemic risk measures.

Keywords conditional systemic risk measure, aggregation function, conditional dual representation, dynamic systemic risk measure, time-consistency

The recent financial crisis emphasizes the influence of systemic risk and the importance of measuring and managing this specific form of risk. Recent research on systemic risk involves

*This work was supported by a fellowship within the Postdoc-Program of the German Academic Exchange Service (DAAD), 91529914.

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several different perspectives on this topic since a given system or network of firms comprises different complex interactions between individual entities and different aspects are interesting for a thorough study. For an overview on existing research see for instance Staum (2013).

In this paper we follow the axiomatic approach to systemic risk. This means we take the viewpoint of a regulator who is interested in risk and stability of the whole system. In context of risk measurement for only one firm, this axiomatic approach was originally introduced by Artzner et al. (1999) who have studied so called coherent risk measures on a finite probability space. The results from Artzner et al. (1999) were extended by Delbaen (2000, 2002) to general probability spaces and by Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002) to the more general class of so called convex risk measures. In case of risk measurement for a whole network of firms, Chen et al. (2013) were the first to consider this axiomatic approach on a finite probability space for so called positively homogeneous systemic risk measures. In Kromer et al. (2014) these results were generalized by studying the case of a general probability space and dropping the axiom of positive homogeneity.

Nevertheless, these systemic risk measures are all static, which means that they do not take into account dynamic features. There exist different possibilities and different leverage points to consider dynamic features. For example one could incorporate the additional information that is available through time or consider processes instead of random variables. In this context, a discrete-time or continuous-time process could represent the equity or asset value process of a specific firm, see Artzner et al. (2007) and Cheridito et al. (2006). For standard risk measures based on the results of Artzner et al. (1999) several approaches study the various aspects of dynamic risk measures, for an overview see for instance Acciaio and Penner (2011).

The aim of this paper is to consider systemic risk measurement in a dynamic framework. We generalize the results from Kromer et al. (2014) and focus on conditional systemic risk measures and dynamic systemic risk measures. The usual approach in the literature to dynamic risk measures is to define these as a family of conditional risk measures with specific time consistency properties. We work with the framework of Cheridito et al. (2006) and Cheridito and Kupper (2011) who consider conditional and dynamic risk measures for bounded discrete-time processes. Moreover, we use the time-consistency concept from Cheridito et al. (2006) and Cheridito and Kupper (2011) and adapt it to our setting of dynamic systemic risk measurement. In particular, we introduce a time-consistency property for our aggregation functions and analyze how the time-consistency of the dynamic systemic risk measure, the dynamic single-firm risk measure and the dynamic aggregation function depend on each other.

The outline of this paper is the following: We start by introducing the general notation in Section 1 and defining conditional convex and conditional positively homogeneous systemic risk measures. In Section 2 we provide a decomposition result for conditional systemic risk measures. Section 3 is dedicated to different representation results. First, we provide a so called primal representation for conditional systemic risk measures. Then we apply techniques from Cheridito et al. (2006) to obtain a dual representation for conditional systemic risk measures. This representation is the main result of this paper. Finally, we study in Section 4 dynamic systemic risk measures as composition of a dynamic single-firm risk measure and dynamic aggregation function. In particular, we discuss the corresponding time-consistency properties.

We also refer to the article Hoffmann et al. (2014) which was written independently of this paper, considers another possible extension of the results in Chen et al. (2013) and Kromer et al.

(2014) to a conditional setting and focuses on the decomposition result only.

1 Notation and definitions

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{N}_0}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and we consider a network of n firms. In our setting the n-dimensional discrete-time process $\bar{X} = (\bar{X}^1, \dots, \bar{X}^n)$ describes the loss processes of the n firms in the underlying network.

For each $m \in \mathbb{N}$, define $\mathcal{R}^{0,m}$ as the space of $\mathbb{F}$-adapted m-dimensional processes. In this paper we understand equalities and inequalities between random variables and stochastic processes $\mathbb{P}$-a.s. Define

$$\mathcal{R}^{\infty,m} := \{\bar{X} \in \mathcal{R}^{0,m} | \|\bar{X}\|_{\mathcal{R}^{\infty,m}} < \infty\} \quad \text{and} \quad \mathcal{A}^{1,m} := \{\bar{\xi} \in \mathcal{R}^{0,m} | \|\bar{\xi}\|_{\mathcal{A}^{1,m}} < \infty\}$$

with the corresponding norms

$$\|\bar{X}\|_{\mathcal{R}^{\infty,m}} := \max_{i\in\{1,\dots,m\}} \|\bar{X}^i\|_{\mathcal{R}^\infty} \quad \text{and} \quad \|\bar{\xi}\|_{\mathcal{A}^{1,m}} := \sum_{i=1}^m \|\bar{\xi}^i\|_{\mathcal{A}^1}$$

and set

$$\|X\|_{\mathcal{R}^\infty} := \inf\left\{ r \in \mathbb{R} \,\middle|\, \sup_{t\in\mathbb{N}_0} |X_t| \le r \right\} \quad \text{and}$$

$$\|\xi\|_{\mathcal{A}^1} := \mathbb{E}\left[\sum_{t\in\mathbb{N}_0} |\Delta\xi_t| \right] \quad \text{where } \xi_{-1} := 0,\ \Delta\xi_t := \xi_t - \xi_{t-1}.$$

In case of $m = 1$, we simply write $\mathcal{R}^\infty := \mathcal{R}^{\infty,1}$ and $\mathcal{A}^1 := \mathcal{A}^{1,1}$.

We equip the space $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with the norm $\|\gamma\|_\infty := \operatorname{ess\,sup} |\gamma|$ for $\gamma \in L^\infty$. Moreover, the bilinear form $\langle \cdot, \cdot \rangle_m : \mathcal{R}^{\infty,m} \times \mathcal{A}^{1,m} \to \mathbb{R}$ is defined by

$$\langle \bar{X}, \bar{\xi} \rangle_m := \sum_{i=1}^m \mathbb{E}\left[\sum_{t\in\mathbb{N}_0} \bar{X}^i_t \Delta\bar{\xi}^i_t \right].$$

Again, we simplify $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_1$.

Remark 1.1. Define the σ-algebra $\mathcal{H}$ on $\mathbb{N}_0 \times \Omega$ as the σ-algebra generated by the sets $\{t\} \times B$, $t \in \mathbb{N}_0$, $B \in \mathcal{F}_t$ and define the measure η on $(\mathbb{N}_0 \times \Omega, \mathcal{H})$ by

$$\eta(\{t\} \times B) := 2^{-(t+1)} \mathbb{P}[B].$$

Cheridito et al. (2006) point out in proof of Lemma 3.17 that $\mathcal{R}^\infty = L^\infty_\mathcal{H} := L^\infty(\mathbb{N}_0 \times \Omega, \mathcal{H}, \eta)$ and $\mathcal{A}^1$ can be identified with $L^1_\mathcal{H} := L^1(\mathbb{N}_0 \times \Omega, \mathcal{H}, \eta)$. The natural pairing $\langle \cdot, \cdot \rangle_{L^\infty_\mathcal{H}, L^1_\mathcal{H}} : L^\infty_\mathcal{H} \times L^1_\mathcal{H} \to \mathbb{R}$ is given by $\langle X, \Xi \rangle_{L^\infty_\mathcal{H}, L^1_\mathcal{H}} := \int X \Xi d\eta$. Similarly, $\mathcal{R}^{\infty,m} = (L^\infty_\mathcal{H})^m$ and $\mathcal{A}^{1,m}$ can be identified with $(L^1_\mathcal{H})^m$. The spaces $(L^\infty_\mathcal{H})^m$ and $(L^1_\mathcal{H})^m$ are endowed with the norms $\|\bar{X}\|_{(L^\infty_\mathcal{H})^m} := \max_{i\in\{1,\dots,m\}} \|\bar{X}^i\|_{L^\infty_\mathcal{H}}$ for $\bar{X} \in (L^\infty_\mathcal{H})^m$ and $\|\bar{\Xi}\|_{(L^1_\mathcal{H})^m} := \sum_{i=1}^m \|\bar{\Xi}^i\|_{L^1_\mathcal{H}}$ for $\bar{\Xi} \in (L^1_\mathcal{H})^m$ and the pairing $\langle \cdot, \cdot \rangle_{(L^\infty_\mathcal{H})^m, (L^1_\mathcal{H})^m} : (L^\infty_\mathcal{H})^m \times (L^1_\mathcal{H})^m \to \mathbb{R}$ is defined by $\langle \bar{X}, \bar{\Xi} \rangle_{(L^\infty_\mathcal{H})^m, (L^1_\mathcal{H})^m} := \sum_{i=1}^m \int \bar{X}^i \bar{\Xi}^i d\eta$.

Because of the previous remark, we can define $\sigma(\mathcal{R}^{\infty,m}, \mathcal{A}^{1,m})$ as the coarest topology on $\mathcal{R}^{\infty,m}$ such that for all $\bar{\xi} \in \mathcal{A}^{1,m}$, $\bar{X} \mapsto \langle \bar{X}, \bar{\xi} \rangle_m$ on $\mathcal{R}^{\infty,m}$ is continuous and linear.

Let τ and θ stopping times with $\tau < \infty$ and $0 \leq \tau \leq \theta \leq \infty$. Define the projection $p_m^{\tau,\theta} : \mathcal{R}^{0,m} \to \mathcal{R}^{0,m}$ by

$$p_m^{\tau,\theta}(\bar{X})_t := I_{\{\tau \leq t\}} \bar{X}_{t \wedge \theta}$$

and the spaces $\mathcal{R}^{\infty,m}_{\tau,\theta} \subset \mathcal{R}^{\infty,m}$ and $\mathcal{A}^{1,m}_{\tau,\theta}$ by

$$\mathcal{R}^{\infty,m}_{\tau,\theta} := p_m^{\tau,\theta} \mathcal{R}^{\infty,m} \quad \text{and} \quad \mathcal{A}^{1,m}_{\tau,\theta} := p_m^{\tau,\theta} \mathcal{A}^{1,m}.$$

Finally, note that for $\bar{\gamma} \in (L^\infty_\tau)^m$ and $\bar{X} \in \mathcal{R}^{\infty,m}$ the processes $\bar{Y} = \bar{\gamma} I_{[\tau,\theta)}$ and $\bar{Z} = \bar{X} I_{[\tau,\theta)}$ are given by

$$\begin{aligned} &\bar{Y}^i_t(\omega) = \bar{\gamma}^i(\omega) I_{[\tau,\theta)}(t,\omega) \quad \text{and} \\ &\bar{Z}^i_t(\omega) = \bar{X}^i_t(\omega) I_{[\tau,\theta)}(t,\omega) \quad \text{for } t \in \mathbb{N}_0, \omega \in \Omega, i \in \{1, \ldots, n\}. \end{aligned}$$

In the following, we consider important properties of $\rho_0 : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$:

(r1) Monotonicity: If $X \geq Y$, then $\rho_0(X) \geq \rho_0(Y)$ for any $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$.

(r2) $\mathcal{F}_\tau$-convexity: $\rho_0(\gamma X + (1-\gamma) Y) \leq \gamma \rho_0(X) + (1-\gamma) \rho_0(Y)$ for any $X, Y \in \mathcal{R}^\infty_{\tau,\theta}$ and any $\gamma \in L^\infty_\tau$ with $\gamma \in [0,1]$.

(r3) $\mathcal{F}_\tau$-translation property: $\rho_0(X + \gamma I_{[\tau,\infty)}) = \rho_0(X) + \gamma$ for any $X \in \mathcal{R}^\infty_{\tau,\theta}$ and any $\gamma \in L^\infty_\tau$.

(r4) $\mathcal{F}_\tau$-positive homogeneity: $\rho_0(\gamma X) = \gamma \rho_0(X)$ for any $X \in \mathcal{R}^\infty_{\tau,\theta}$ and any $\gamma \in (L^\infty_\tau)_+$.

(r5) Constancy on $\mathfrak{R} \subset L^\infty_\tau$: $\rho_0(\gamma I_{[\tau,\infty)}) = \gamma$ for any $\gamma \in \mathfrak{R}$.

(r6) Normalization: $\rho_0(I_{[\tau,\infty)}) = 1$.

These properties are analogous to the properties of ρ_0 from Kromer et al. (2014) and the interpretation is the same.

Definition 1.2. *A* conditional convex single-firm risk measure (on $\mathcal{R}^\infty_{\tau,\theta}$) *is a function* $\rho_0 : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ *that satisfies the properties (r1) and (r2). A* conditional positively homogeneous single-firm risk measure (on $\mathcal{R}^\infty_{\tau,\theta}$) *is a conditional convex single-firm risk measure (on* $\mathcal{R}^\infty_{\tau,\theta}$*) that additionally satisfies the properties (r4) and (r6). A* conditional coherent single-firm risk measure (on $\mathcal{R}^\infty_{\tau,\theta}$) *is a conditional positively homogeneous single-firm risk measure (on* $\mathcal{R}^\infty_{\tau,\theta}$*) that additionally satisfies the property (r3). Moreover, for* $X \in \mathcal{R}^\infty$ *we define* $\rho_0(X) := \rho_0(p^{\tau,\theta}(X))$.

If additionally $\rho_0(0) = 0$ is satisfied, then the translation property (r3) is stronger than constancy property on L^∞_τ.

Now, consider a map $\rho : \mathcal{R}^{\infty,n}_{\tau,\theta} \to L^\infty_\tau$ that possibly satisfies the following properties:

(s1) Monotonicity: If $\bar{X} \geq \bar{Y}$, then $\rho(\bar{X}) \geq \rho(\bar{Y})$ for any $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$.

(s2) Preference consistency: If $\rho(\bar{X}_t(\omega) I_{[\tau,\infty)}) \geq \rho(\bar{Y}_t(\omega) I_{[\tau,\infty)})$ for all $t \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$, then $\rho(\bar{X}) \geq \rho(\bar{Y})$.

(s3) f_ρ-constancy: Either $\operatorname{Im} \rho|_{\mathbb{R}^n I_{[\tau,\infty)}} = \mathbb{R}$ and there exists a surjective map $f_\rho : \mathbb{R} \to \mathbb{R}$ with $f_\rho(0) = 0$ such that $\rho(a 1_n I_{[\tau,\infty)}) = f_\rho(a)$ for any $a \in \mathbb{R}$ or $\operatorname{Im} \rho|_{\mathbb{R}^n I_{[\tau,\infty)}} = \mathbb{R}_+$ and there exists a map $f_\rho : \mathbb{R} \to \mathbb{R}_+$ and $b \in \mathbb{R}_+$ such that f_ρ is surjective and strictly increasing on $[b, \infty)$, $f_\rho(a) = 0$ for $a \leq b$ and $\rho(a 1_n I_{[\tau,\infty)}) = f_\rho(a)$ for any $a \in \mathbb{R}$.

(s4) $\mathcal{F}_\tau$-convexity:

(s5) $\mathcal{F}_\tau$-positive homogeneity: $\rho(\gamma \bar{X}) = \gamma \rho(\bar{X})$ for any $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ and any $\gamma \in (L^\infty_\tau)_+$.

(s6) Normalization: $\rho(1_n I_{[\tau,\infty)}) = n$.

(s3) differs from property (S3) in Kromer et al. (2014) by the additional assumption that $f_\rho(0) = 0$ is satisfied. This property is needed because of the close connection between the systemic risk measure ρ and the corresponding aggregation function Λ. This aggregation function has to guarantee that the image of every $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ satisfies $\Lambda(\bar{X}) = \Lambda(\bar{X}) I_{[\tau,\infty)}$. Hence, before time τ the process $\Lambda(\bar{X})$ is equal to zero. Later on, we will prove that if $f_\Lambda(0) = 0$ is satisfied then Λ admits this property. Moreover, as in Kromer et al. (2014) there exists an inverse function f_ρ^{-1} of f_ρ.

Definition 1.3. *A* conditional positively homogeneous systemic risk measure (on $\mathcal{R}^{\infty,n}_{\tau,\theta}$) *is a function $\rho : \mathcal{R}^{\infty,n}_{\tau,\theta} \to L^\infty_\tau$ that satisfies the properties (s1), (s2), (s4), (s5), (s6) and* $\operatorname{Im} \rho|_{\mathbb{R}^n I_{[\tau,\infty)}} \subset \mathbb{R}$. *If ρ satisfies the properties (s1)-(s4), we will call ρ a* conditional convex systemic risk measure (on $\mathcal{R}^{\infty,n}_{\tau,\theta}$). *Moreover, for $\bar{X} \in \mathcal{R}^{\infty,n}$ we define $\rho(\bar{X}) := \rho(p^{\tau,\theta}_n(\bar{X}))$.*

For the remaining part of this section we consider aggregation functions. In Kromer et al. (2014) convex aggregation functions are defined as functions from $\mathbb{R}^n$ to $\mathbb{R}$. We use a similar definition in our setting. Consider the following properties of a function $\Lambda : \mathbb{R}^n \to \mathbb{R}$:

(a1) Monotonicity: If $\bar{x} \geq \bar{y}$, then $\Lambda(\bar{x}) \geq \Lambda(\bar{y})$ for any $\bar{x}, \bar{y} \in \mathbb{R}^n$.

(a2) Convexity: $\Lambda(a\bar{x} + (1-a)\bar{y}) \leq a\Lambda(\bar{x}) + (1-a)\Lambda(\bar{y})$ for any $\bar{x}, \bar{y} \in \mathbb{R}^n$ and any $a \in [0,1]$.

(a3) f_Λ-constancy: Either $\operatorname{Im} \Lambda = \mathbb{R}$ and there exists a surjective map $f_\Lambda : \mathbb{R} \to \mathbb{R}$ with $f_\Lambda(0) = 0$ such that $\Lambda(a1_n) = f_\Lambda(a)$ for any $a \in \mathbb{R}$ or $\operatorname{Im} \Lambda = \mathbb{R}_+$ and there exists a map $f_\Lambda : \mathbb{R} \to \mathbb{R}_+$ and $b \in \mathbb{R}_+$ such that f_Λ is surjective and strictly increasing on $[b, \infty)$, $f_\Lambda(a) = 0$ for $a \leq b$ and $\Lambda(a1_n) = f_\Lambda(a)$ for any $a \in \mathbb{R}$.

(a4) Positive homogeneity: $\Lambda(a\bar{x}) = a\Lambda(\bar{x})$ for any $\bar{x} \in \mathbb{R}^n$ and any $a \in \mathbb{R}_+$.

(a5) Normalization: $\Lambda(1_n) = n$.

In comparison to (A1)-(A5) in Kromer et al. (2014), the only difference is the additional assumption $f_\Lambda(0) = 0$ in the f_Λ-constancy property.

Definition 1.4. *A* positively homogeneous aggregation function *is a function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ that satisfies the properties (a1), (a2), (a4) and (a5). If Λ satisfies the properties (a1)-(a3) then we will call Λ a* convex aggregation function.

First, note that every convex aggregation function Λ is continuous since convex and finite valued functions on $\mathbb{R}^n$ are continuous. Hence, Λ is measurable. Now, consider $\bar{X} \in \mathcal{R}^{\infty,m}$ as function $\bar{X} : (\mathbb{N}_0 \times \Omega, \mathcal{H}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and Λ as function $\Lambda : (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. It follows that $\Lambda(\bar{X}) := \Lambda \circ \bar{X}$ is also measurable and $\Lambda(\bar{X}) : (\mathbb{N}_0 \times \Omega, \mathcal{H}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfies $\Lambda(\bar{X}_t(\omega)) = \Lambda(\bar{X})_t(\omega)$ for all $t \in \mathbb{N}_0$ and all $\omega \in \Omega$.

Our next aim is to prove that every aggregation function induces a map $\Lambda : \mathcal{R}^{\infty,n}_{\tau,\theta} \to \mathcal{R}^{\infty}_{\tau,\theta}$.

Lemma 1.5. *For an increasing function $g : \mathbb{R} \to \mathbb{R}$ we have $g(\mathcal{R}^\infty) \subset \mathcal{R}^\infty$. If additionally $g(0) = 0$, then $g(\mathcal{R}^\infty_{\tau,\theta}) \subset \mathcal{R}^\infty_{\tau,\theta}$.*

Proof. The first assertion follows directly from the definition of $\mathcal{R}^\infty$ and the monotonicity of g. Moreover, if additionally $g(0) = 0$ is satisfied, then we directly obtain $g(\mathcal{R}^\infty_{\tau,\theta}) \subset \mathcal{R}^\infty_{\tau,\theta}$. □

The following result is based on Lemma 2.4 in Kromer et al. (2014). Moreover, note that $f_\Lambda(0) = 0$ guarantees that for every $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ we have $\Lambda(\bar{X}) \in \mathcal{R}^\infty_{\tau,\theta}$.

Lemma 1.6. *Let $\Lambda : \mathbb{R}^n \to \mathbb{R}$ be a convex aggregation function and $\operatorname{Im}\Lambda = \mathbb{R}$ [respectively $\operatorname{Im}\Lambda = \mathbb{R}_+$]. Then $\Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = \mathcal{R}^\infty_{\tau,\theta}$ [respectively $\Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = (\mathcal{R}^\infty_{\tau,\theta})_+$].*

Proof. Let $\operatorname{Im}\Lambda = \mathbb{R}$. The inclusion $\Lambda(\mathcal{R}^{\infty,n}) \subset \mathcal{R}^\infty$ follows directly from Lemma 2.4 in Kromer et al. (2014) and the fact that $\mathcal{R}^{\infty,n} = (L^\infty_\mathcal{H})^n$. $\Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) \subset \mathcal{R}^\infty_{\tau,\theta}$ is guaranteed by $\Lambda(0 \cdot 1_n) = f_\Lambda(0) = 0$.

To show the other inclusion, consider $X \in \mathcal{R}^\infty_{\tau,\theta}$. As in the proof of Lemma 2.4 in Kromer et al. (2014), we can define a process

$$Y^X_t(\omega) := f_\Lambda^{-1}(X_t(\omega)) \quad \text{for } t \in \mathbb{N}_0 \text{ and } \omega \in \Omega. \tag{1}$$

Lemma 1.5 yields that $Y^X = f_\Lambda^{-1}(X) \in \mathcal{R}^\infty_{\tau,\theta}$. Moreover, by definition of Y^X, we obtain $f_\Lambda(Y^X) = X$. The f_Λ-constancy property finally implies $\Lambda(Y^X 1_n) = f_\Lambda(Y^X) = X$, which means that $\mathcal{R}^\infty_{\tau,\theta} \subset \Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta})$

In case of $\operatorname{Im}\Lambda = \mathbb{R}_+$, we have $f_\Lambda^{-1} : \mathbb{R}_+ \to [b, \infty)$ and $f_\Lambda^{-1}(0) = b$ and the process defined in (1) satisfies $Y^X \in \mathcal{R}^\infty_{0,\theta}$ for $X \in (\mathcal{R}^\infty_{\tau,\theta})_+$. Nevertheless, the process $(Y^X)'$ defined by $(Y^X)' := Y^X I_{[\tau,\infty)}$ satisfies $(Y^X)' \in \mathcal{R}^\infty_{\tau,\theta}$ and $f_\Lambda(((Y^X)')) = X$. □

2 Structural decomposition

The aim of this section is the extension of the decomposition results in Kromer et al. (2014) to conditional systemic risk measures. The proofs are similar to the proofs in Kromer et al. (2014). Hence, we repeat only the main ideas.

Theorem 2.1. *(a) A function $\rho : \mathcal{R}^{\infty,n}_{\tau,\theta} \to L^\infty_\tau$ with $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}$ is a conditional convex systemic risk measure if and only if there exists a convex aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ and $\Lambda(\mathbb{R}^n) = \mathbb{R}$ and a conditional convex single-firm risk measure $\rho_0 : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ that satisfies the constancy property on $\mathbb{R}$ such that ρ is the composition of ρ_0 and Λ, i.e.*

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \textit{for all } \bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}.$$

(b) *A function* $\rho : \mathcal{R}^{\infty,n}_{\tau,\theta} \to L^{\infty}_{\tau}$ *with* $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}_+$ *is a conditional convex systemic risk measure if and only if there exists a convex aggregation function* $\Lambda : \mathbb{R}^n \to \mathbb{R}$ *with* $\Lambda(\mathbb{R}^n) = \mathbb{R}_+$ *and a conditional convex single-firm risk measure* $\rho_0 : \mathcal{R}^{\infty}_{\tau,\theta} \to L^{\infty}_{\tau}$ *that satisfies the constancy property on* $\mathbb{R}_+$ *such that* ρ *is the composition of* ρ_0 *and* Λ*, i.e.*

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \textit{for all } \bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}.$$

Proof. In case of (a), we define $\mathfrak{R} = \mathfrak{S} = \mathbb{R}$ and in case of (b), we define $\mathfrak{R} = \mathbb{R}_+$ and $\mathfrak{S} = [b, \infty)$ for $b \in \mathbb{R}_+$. If ρ is a conditional convex systemic risk measure with $f_\rho : \mathbb{R} \to \mathfrak{R}$ that is surjective and strictly increasing on $\mathfrak{S}$, define the aggregation function Λ by

$$\Lambda(\bar{x}) := \rho(\bar{x} I_{[\tau,\infty)}) \quad \text{for } \bar{x} \in \mathbb{R}^n.$$

Then convexity (a2), monotonicity (a1) and f_Λ-constancy follows from the corresponding properties of ρ. Furthermore, $\Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = \mathcal{R}^{\infty}_{\tau,\theta}$ in case of (a) and $\Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = (\mathcal{R}^{\infty}_{\tau,\theta})_+$ in case of (b) follows from Lemma 1.6.

The conditional single-firm risk measure $\rho_0 : \mathcal{R}^{\infty}_{\tau,\theta} \to L^{\infty}_{\tau}$ is defined by

$$\rho_0(X) := \begin{cases} \tilde{\rho}_0(X) & \text{if } \Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = \mathcal{R}^{\infty}_{\tau,\theta} \\ \tilde{\rho}_0(X^+) & \text{if } \Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) = (\mathcal{R}^{\infty}_{\tau,\theta})_+ \end{cases}$$

where the stochastic process $X^+ \in (\mathcal{R}^{\infty}_{\tau,\theta})_+$ is given by $X^+_t(\omega) := \max\{X_t(\omega), 0\}$ for $t \in \mathbb{N}_0$ and $\omega \in \Omega$ and $\tilde{\rho}_0 : \Lambda(\mathcal{R}^{\infty,n}_{\tau,\theta}) \to L^{\infty}_{\tau}$ is defined by

$$\tilde{\rho}_0\,(X) := \rho(\bar{X}) \quad \text{where } \bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta} \text{ satisfies } \Lambda(\bar{X}) = X.$$

Then $\tilde{\rho}_0$ is well-defined because of the preference consistency of ρ. Monotonicity, convexity and constancy on $\mathfrak{R}$ follow as in Kromer et al. (2014). Moreover, ρ_0 inherits these properties from $\tilde{\rho}_0$. Finally, the definition of Λ and ρ_0 implies $\rho = \rho_0 \circ \Lambda$.

The proof of the second part is analogous to the proof in Kromer et al. (2014). □

For the proof of the positively homogeneous case below see Corollary 3.3 in Kromer et al. (2014).

Corollary 2.2. (a) *A function* $\rho : \mathcal{R}^{\infty,n}_{\tau,\theta} \to L^{\infty}_{\tau}$ *with* $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}$ *is a conditional positively homogeneous systemic risk measure if and only if there exists a positively homogeneous aggregation function* $\Lambda : \mathbb{R}^n \to \mathbb{R}$ *and* $\Lambda(\mathbb{R}^n) = \mathbb{R}$ *and a conditional coherent single-firm risk measure* $\rho_0 : \mathcal{R}^{\infty}_{\tau,\theta} \to L^{\infty}_{\tau}$ *such that* ρ *is the composition of* ρ_0 *and* Λ*, i.e.*

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \textit{for all } \bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}.$$

(b) *A function* $\rho : \mathcal{R}^{\infty,n}_{\tau,\theta} \to L^{\infty}_{\tau}$ *with* $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}_+$ *is a conditional positively homogeneous systemic risk measure if and only if there exists a positively homogeneous aggregation function* $\Lambda : \mathbb{R}^n \to \mathbb{R}$ *with* $\Lambda(\mathbb{R}^n) = \mathbb{R}_+$ *and a conditional positively homogeneous single-firm risk measure* $\rho_0 : \mathcal{R}^{\infty}_{\tau,\theta} \to L^{\infty}_{\tau}$ *such that* ρ *is the composition of* ρ_0 *and* Λ*, i.e.*

$$\rho(\bar{X}) = (\rho_0 \circ \Lambda)(\bar{X}) \quad \textit{for all } \bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}.$$

3 Representations of conditional systemic risk measures

3.1 Primal representation

In this subsection we provide the so called primal representation of conditional convex risk measures ρ. From now on, let $\rho = \rho_0 \circ \Lambda$ where ρ_0 is the corresponding conditional convex single-firm risk measure and Λ is the corresponding convex aggregation function. It follows that $\operatorname{Im}\Lambda = \mathbb{R}$ or $\operatorname{Im}\Lambda = \mathbb{R}_+$. Moreover, $\rho_0 : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ satisfies the constancy property on $\mathbb{R}$ if $\operatorname{Im}\Lambda = \mathbb{R}$ and the constancy property on $\mathbb{R}_+$ if $\operatorname{Im}\Lambda = \mathbb{R}_+$. Consider the *acceptance sets* defined by

$$\tilde{\mathcal{B}}_{\rho_0} := \{(\gamma, X) \in L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta} | \gamma \geq \rho_0(X)\} \quad \text{and} \quad \mathcal{B}_\Lambda := \{(Y, \bar{X}) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta} | Y \geq \Lambda(\bar{X})\}.$$

The following definition equals Definition 4.1 in Kromer et al. (2014) for subsets of $L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta}$ and $\mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta}$.

Definition 3.1. *Let $V \times W$ be equal to $L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta}$ or equal to $\mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta}$. A set $\mathcal{S} \subset V \times W$ satisfies the* monotonicity *property if $(v, w_1) \in \mathcal{S}$, $w_2 \in W$ and $w_1 \geq w_2$ implies $(v, w_2) \in \mathcal{S}$. A set $\mathcal{S} \subseteq V \times W$ satisfies the* epigraph *property if $(v_1, w) \in \mathcal{S}$, $v_2 \in V$ and $v_2 \geq v_1$ implies $(v_2, w) \in \mathcal{S}$.*

Proposition 3.2. *Suppose that $\rho = \rho_0 \circ \Lambda$ is a conditional convex systemic risk measure with conditional convex single-firm risk measure $\rho_0 : \mathcal{R}^\infty_{\tau,\theta} \to L^\infty_\tau$ and convex aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$. Let $\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_\Lambda$ be the corresponding acceptance sets.*

1. *$\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_\Lambda$ satisfy the following properties:*

   (a) *If ρ_0 and Λ satisfy the normalization property $\rho_0(1I_{[\tau,\infty)}) = 1$ and $\Lambda(1_n) = n$, then $(1, 1I_{[\tau,\infty)}) \in \tilde{\mathcal{B}}_{\rho_0}$ and $(nI_{[\tau,\infty)}, 1_n I_{[\tau,\infty)}) \in \mathcal{B}_\Lambda$ with $\operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | (\gamma, 1I_{[\tau,\infty)}) \in \tilde{\mathcal{B}}_{\rho_0}\} = 1$ and $\operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | (\gamma I_{[\tau,\infty)}, 1_n I_{[\tau,\infty)}) \in \mathcal{B}_\Lambda\} = n$.*

   (b) *$\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_\Lambda$ satisfy the monotonicity property.*

   (c) *$\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_\Lambda$ satisfy the epigraph property.*

   (d) *$\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_\Lambda$ are $\mathcal{F}_\tau$-convex sets.*

   If ρ is a conditional positively homogeneous systemic risk measure, then

   e) *$\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_\Lambda$ are $\mathcal{F}_\tau$-cones.*

2. *ρ admits for all $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ the so called* primal representation

$$\rho(\bar{X}) = \operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | (\gamma, Y) \in \tilde{\mathcal{B}}_{\rho_0}, (Y, \bar{X}) \in \mathcal{B}_\Lambda\} \tag{2}$$

   where we set $\operatorname{ess\,inf}\emptyset := +\infty$.

Proof. Part 1.a) is trivial and the properties of ρ_0 and Λ directly imply properties 1.b)-1.e) for $\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_\Lambda$. Moreover, ρ_0 satisfies for all $X \in \mathcal{R}^\infty_{\tau,\theta}$

$$\rho_0(X) = \operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | \gamma \geq \rho_0(X)\} = \operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | (\gamma, X) \in \tilde{\mathcal{B}}_{\rho_0}\}.$$

Since $\mathcal{R}^\infty = L^\infty_{\mathcal{H}}$, we can consider $V \in \mathcal{R}^\infty_{\tau,\theta}$ as element in $L^\infty_{\mathcal{H}}$. We obtain

$$\begin{aligned}\Lambda(\bar{Z}) &= \operatorname{ess\,inf}\{V \in L^\infty_{\mathcal{H}} | V \geq \Lambda(\bar{Z})\} = \operatorname{ess\,inf}\{V \in L^\infty_{\mathcal{H}} | V \geq \Lambda(\bar{Z}), V \in \mathcal{R}^\infty_{\tau,\theta}\} \\ &= \operatorname{ess\,inf}\{V \in L^\infty_{\mathcal{H}} | (V, \bar{Z}) \in \mathcal{B}_\Lambda\}.\end{aligned}$$

Since $\rho = \rho_0 \circ \Lambda$, it follows for all $\bar{Z} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ that

$$\begin{aligned}\rho(\bar{Z}) &= \operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | \gamma \geq (\rho_0 \circ \Lambda)(\bar{Z})\} = \operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | (\gamma, \Lambda(\bar{Z})) \in \tilde{\mathcal{B}}_{\rho_0}\} \\ &= \operatorname{ess\,inf}\{\gamma \in L^\infty_\tau | (\gamma, \operatorname{ess\,inf}\{V \in L^\infty_{\mathcal{H}} | (V, \bar{Z}) \in \mathcal{B}_\Lambda\}) \in \tilde{\mathcal{B}}_{\rho_0}\}\end{aligned}$$

Finally, the monotonicity of $\tilde{\mathcal{B}}_{\rho_0}$ yields (2). □

3.2 Continuity and closedness

This subsection provides the basic results to prove a dual representation for conditional convex systemic risk measures $\rho = \rho_0 \circ \Lambda$. Note that any $\gamma \in L^\infty_\tau$ we can be identified by $\gamma I_{[\tau,\infty)} \in \mathcal{R}^\infty_{\tau,\theta}$. Thus, there exists a one to one relation between $\tilde{\mathcal{B}}_{\rho_0}$ and $\mathcal{B}_{\rho_0}$ where $\mathcal{B}_{\rho_0}$ is defined by

$$\mathcal{B}_{\rho_0} := \{(X, Z) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta} | X = \gamma I_{[\tau,\infty)} \text{ for } \gamma \in L^\infty_\tau, \gamma \geq \rho_0(Z)\}.$$

In the following lemma we generalize the results from Lemma A.65 in Föllmer and Schied (2011) for multidimensional spaces.

Lemma 3.3. *Define the set* $\mathcal{B}^r_{\infty,m} := \{\bar{X} \in (L^\infty_{\mathcal{H}})^m | \|\bar{X}\|_{(L^\infty_{\mathcal{H}})^m} \leq r\}$ *for* $r > 0$.

1. *For every* $r > 0$ *the set* $\mathcal{B}^r_{\infty,m}$ *is closed in* $(L^1_{\mathcal{H}})^m$.
2. *A convex subset* $\mathcal{C}$ *of* $(L^\infty_{\mathcal{H}})^m$ *is* $\sigma((L^\infty_{\mathcal{H}})^m, (L^1_{\mathcal{H}})^m)$*-closed if for every* $r > 0$ *the set* $\mathcal{C}_r := \mathcal{C} \cap \mathcal{B}^r_{\infty,m}$ *is closed in* $(L^1_{\mathcal{H}})^m$.

Proof. 1. For every sequence $(\bar{Y}^{(k)}) \subset \mathcal{B}^r_{\infty,m}$ with $\bar{Y}^{(k)} \to \bar{Y}$ in $(L^1_{\mathcal{H}})^m$ we can find a subsequence $\bar{Y}^{(k_l)}$ with $(\bar{Y}^{(k_l)})^i \to (\bar{Y})^i$ η-a.s. and in $L^1_{\mathcal{H}}$ for each $i \in \{1, \ldots, m\}$. Since

$$|\bar{Y}^i| \leq |(\bar{Y}^{(k_l)})^i| + |(\bar{Y}^{(k_l)})^i - \bar{Y}^i| \leq \|(\bar{Y}^{(k_l)})^i\|_{L^\infty_{\mathcal{H}}} + |(\bar{Y}^{(k_l)})^i - \bar{Y}^i|,$$

we obtain $\max_{i\in\{1,\ldots,m\}} |\bar{Y}^i| \leq r + \max_{i\in\{1,\ldots,m\}} |\bar{Y}^{(k_l)})^i - \bar{Y}^i|$ η-a.s. Since the convergence $(\bar{Y}^{(k_l)})^i) \to \bar{Y}^i$ holds η-a.s., it follows $\max_{i\in\{1,\ldots,m\}} |\bar{Y}^i| \leq r$ η-a.s. This implies $|\bar{Y}^i| \leq r$ η-a.s. for any $i \in \{1, \ldots, m\}$ and we get $\|\bar{Y}^i\|_{L^\infty_{\mathcal{H}}} \leq r$ η-a.s. for any $i \in \{1, \ldots, m\}$. Finally this leads to $\max_{i\in\{1,\ldots,m\}} \|\bar{Y}^i\|_{L^\infty_{\mathcal{H}}} \leq r$ which implies $\bar{Y} \in \mathcal{B}^r_{\infty,m}$.

2. See Lemma A.65 in Föllmer and Schied (2011). □

The following definition is based on Definition 3.15 in Cheridito et al. (2006).

Definition 3.4. *We say that a sequence* $(\bar{X}^{(k)}) \subset \mathcal{R}^{\infty,m}_{\tau,\theta}$ *is* increasing (decreasing) *if each* $(\bar{X}^{(k)})^i \subset \mathcal{R}^\infty_{\tau,\theta}$*,* $i \in \{1, \ldots, m\}$*, is increasing (decreasing) in* k*. Moreover,* $\bar{X}^{(k)}_s \uparrow \bar{X}_s$ $\mathbb{P}$*-a.s. for some* $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$ *and* $s \in \mathbb{N}_0$*, if* $(\bar{X}^{(k)}_s)^i \uparrow (\bar{X}_s)^i$ $\mathbb{P}$*-a.s. for all* $i \in \{1, \ldots, m\}$*.*

We call a map $\upsilon : \mathcal{R}^{\infty,m}_{\tau,\theta} \to L^\infty$ continuous for bounded increasing sequences *if for* $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$ *and every increasing sequence* $(\bar{X}^{(k)}) \subset \mathcal{R}^{\infty,m}_{\tau,\theta}$ *with* $\bar{X}^{(k)}_t \uparrow \bar{X}_t$ $\mathbb{P}$*-a.s. for all* $t \in \mathbb{N}_0$*, we obtain* $\lim_{k\to\infty} \upsilon(\bar{X}^{(k)}) = \upsilon(\bar{X})$ $\mathbb{P}$*-a.s.*

Similarly, a function $\Upsilon : \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,m}_{\tau,\theta} \to L^\infty$ *is called* continuous for bounded decreasing sequences in the first argument and bounded increasing sequences in the second argument *if for* $(X, \bar{Y}) \in \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,m}_{\tau,\theta}$ *and every sequence* $(X^{(k)}, \bar{Y}^{(k)}) \subset \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty,m}_{\tau,\theta}$ *that satisfies* $X^{(k)}_t \downarrow X_t$ *and* $\bar{Y}^{(k)}_t \uparrow \bar{Y}_t$ $\mathbb{P}$*-a.s. for all* $t \in \mathbb{N}_0$*, we obtain* $\lim_{k\to\infty} \Upsilon(X^{(k)}, \bar{Y}^{(k)}) = \Upsilon(X, \bar{Y})$ $\mathbb{P}$*-a.s.*

In the following lemma we study the closedness of the acceptance set $\mathcal{B}_{\rho_0}$. The proof is similar the proof of Lemma 3.17 in Cheridito et al. (2006).

Lemma 3.5. *Let* ρ_0 *be a conditional convex single-firm risk measure. If* ρ_0 *is continuous for bounded increasing sequences, then* $\mathcal{B}_{\rho_0}$ *is* $\sigma(\mathcal{R}^\infty \times \mathcal{R}^\infty, \mathcal{A}^1 \times \mathcal{A}^1)$*-closed.*

Proof. Define $\varrho : \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty}_{\tau,\theta} \to L^\infty_\tau$, $\varrho(X, Y) := \rho_0(Y) - X_\tau$. Then ϱ is continuous for bounded decreasing sequences in the first argument and bounded increasing sequences in the second argument and

$$\mathcal{B}_{\rho_0} = \{(X, Z) \in \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty}_{\tau,\theta} | X = \gamma I_{[\tau,\infty)} \text{ for } \gamma \in L^\infty_\tau \text{ and } 0 \geq \varrho(\gamma I_{[\tau,\infty)}, Z)\}.$$

Let $(X^\iota, Y^\iota) \subset \mathcal{B}_{\rho_0}$ be a net such that $(X^\iota, Y^\iota) \to (X^\#, Y^\#)$ in $(\mathcal{R}^\infty \times \mathcal{R}^\infty, \sigma(\mathcal{R}^\infty \times \mathcal{R}^\infty, \mathcal{A}^1 \times \mathcal{A}^1))$ for $(X^\#, Y^\#) \in \mathcal{R}^\infty$. $(X^\#, Y^\#)$ satisfies $(X^\#, Y^\#) \in \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty}_{\tau,\theta}$ with $X^\# = \gamma^\# I_{[\tau,\infty)}$ for some $\gamma^\# \in L^\infty_\tau$. Suppose that $0 < \varrho(X^\#, Y^\#)$ on $A \in \mathcal{F}_\tau$ with $\mathbb{P}[A] > 0$. Define $\tilde{\varrho} : \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty}_{\tau,\theta} \to \mathbb{R}$ by

$$\tilde{\varrho}(X, Z) := \mathbb{E}[\varrho(X_\tau I_{[\tau,\infty)}, Z) I_A].$$

Then $\tilde{\varrho}$ is decreasing in the first and increasing in the second argument, convex and continuous for bounded decreasing sequences in the first argument and bounded increasing sequences in the second argument.

For the remaining part of the proof we use that $\mathcal{R}^\infty = L^\infty_\mathcal{H}$ and that $\mathcal{A}^1$ can be identified with $L^1_\mathcal{H}$. Consider the set $\mathcal{D}_{\tilde{\varrho}} := \{(X, Z) \in \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty}_{\tau,\theta} | X = \gamma I_{[\tau,\infty)} \text{ for } \gamma \in L^\infty_\tau \text{ and } 0 \geq \tilde{\varrho}(\gamma I_{[\tau,\infty)}, Z)\}$. If we can prove that for each $r > 0$

$$\mathcal{C}_r := \mathcal{D}_{\tilde{\varrho}} \cap \{(X, Z) \in L^\infty_\mathcal{H} \times L^\infty_\mathcal{H} | \, \|(X, Z)\|_{L^\infty_\mathcal{H} \times L^\infty_\mathcal{H}} \leq r\} \subset L^1_\mathcal{H} \times L^1_\mathcal{H}$$

is closed in $L^1_\mathcal{H} \times L^1_\mathcal{H}$, then we obtain by the second part of Lemma 3.3 that $\mathcal{D}_{\tilde{\varrho}}$ is $\sigma(L^\infty_\mathcal{H} \times L^\infty_\mathcal{H}, L^1_\mathcal{H} \times L^1_\mathcal{H})$-closed.

Let $(\gamma^{(k)} I_{[\tau,\infty)}, Z^{(k)}) \subset \mathcal{C}_r$ be a sequence with $(\gamma^{(k)} I_{[\tau,\infty)}, Z^{(k)}) \to (\gamma I_{[\tau,\infty)}, Z)$ in $L^1_\mathcal{H} \times L^1_\mathcal{H}$. $\|(\gamma I_{[\tau,\infty)}, Z)\|_{L^\infty_\mathcal{H} \times L^\infty_\mathcal{H}} \leq r$ follows from the first part of Lemma 3.3 such that we have to prove that $0 \geq \tilde{\varrho}(\gamma I_{[\tau,\infty)}, Z)$. We can find a subsequence $(\gamma^{(k_l)} I_{[\tau,\infty)}, Z^{(k_l)}) \subset L^1_\mathcal{H} \times L^1_\mathcal{H}$ such that $(\gamma^{(k_l)} I_{\{\tau \leq t\}}, Z^{(k_l)}_t) \to (\gamma I_{\{\tau \leq t\}}, Z_t)$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$. Moreover, the sequence $(Y^{(m)})$ defined by $Y^{(m)}_t := \inf_{l \geq m} (Z^{(k_l)}_t \wedge Z_t)$ for all $t \in \mathbb{N}_0$ satisfies $Y^{(m)}_t \uparrow Z_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$ and $Y^{(m)}_t \leq Z^{(k_m)}_t$ for all $m \in \mathbb{N}$ and $t \in \mathbb{N}_0$. Similarly, the sequence $(\phi^{(m)})$ defined by $\phi^{(m)} := \sup_{l \geq m} (\gamma^{(k_l)} \vee \gamma)$ satisfies $\phi^{(m)} \downarrow \gamma$ $\mathbb{P}$-a.s. and $\phi^{(m)} \geq \gamma^{(k_m)}$.

It follows that

$$0 \geq \liminf_{m\to\infty} \tilde{\varrho}(\gamma^{(k_m)} I_{[\tau,\infty)}, Z^{(k_m)}) \geq \lim_{m\to\infty} \tilde{\varrho}(\phi^{(m)} I_{[\tau,\infty)}, Y^{(m)}) = \tilde{\varrho}(\gamma I_{[\tau,\infty)}, Z),$$

which means that $\mathcal{C}_r$ is closed in $L^1_{\mathcal{H}} \times L^1_{\mathcal{H}}$.

Since $0 \geq \varrho(\gamma^\iota I_{[\tau,\infty)}, Y^\iota)$, we have $\tilde{\varrho}(\gamma^\iota I_{[\tau,\infty)}, Y^\iota) = \mathbb{E}[I_A \varrho(\gamma^\iota I_{[\tau,\infty)}, Y^\iota)] \leq 0$. Hence, $(X^\iota, Y^\iota) \in \mathcal{D}_{\tilde{\varrho}}$. The $\sigma(L^\infty_{\mathcal{H}} \times L^\infty_{\mathcal{H}}, L^1_{\mathcal{H}} \times L^1_{\mathcal{H}})$-closedness of $\mathcal{D}_{\tilde{\varrho}}$ implies $(X^\#, Y^\#) \in \mathcal{D}_{\tilde{\varrho}}$ and $\tilde{\varrho}(X^\#, Y^\#) = \mathbb{E}[\varrho(X^\#, Y^\#) I_A] \leq 0$. By the assumptions on ϱ and A, this is a contradiction and therefore, $\mathcal{B}_{\rho_0}$ is $\sigma(\mathcal{R}^\infty \times \mathcal{R}^\infty, \mathcal{A}^1 \times \mathcal{A}^1)$-closed. □

Note that since every convex aggregation function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is continuous, Λ satisfies the following continuity property: For $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$ and every increasing sequence $(\bar{X}^{(k)}) \subset \mathcal{R}^{\infty,m}_{\tau,\theta}$ with $\bar{X}^{(k)}_t \uparrow \bar{X}_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$, we have $\Lambda(\bar{X}^{(k)}_t) \uparrow \Lambda(\bar{X}_t)$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$.

Lemma 3.6. *Let Λ be a convex aggregation function. Then $\mathcal{B}_\Lambda$ is $\sigma(\mathcal{R}^{\infty,n+1}, \mathcal{A}^{1,n+1})$-closed.*

Proof. The map Υ_t defined by $\Upsilon_t(Y, \bar{Z}) := \Lambda(\bar{Z}_t) - Y_t$ for $(Y, \bar{Z}) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta}$ and all $t \in \mathbb{N}_0$ is continuous for bounded decreasing sequences in the first argument and bounded increasing sequences in the second argument. Moreover, we have

$$\mathcal{B}_\Lambda = \{(X, \bar{Z}) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta} | 0 \geq \Upsilon_t(X, \bar{Z}) \text{ for all } t \in \mathbb{N}_0\}.$$

Let $(Y^\iota, \bar{X}^\iota) \subset \mathcal{B}_\Lambda$ be a net such that $(Y^\iota, \bar{X}^\iota) \to (Y^\#, \bar{X}^\#)$ in $(\mathcal{R}^{\infty,n+1}, \sigma(\mathcal{R}^{\infty,n+1}, \mathcal{A}^{1,n+1}))$ for and $(Y^\#, \bar{X}^\#) \in \mathcal{R}^{\infty,n+1}$. It follows directly that $(Y^\#, \bar{X}^\#) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta}$. Suppose that there exists $s \in \mathbb{N}_0$ such that $0 < \Upsilon_s(Y^\#, \bar{X}^\#)$ on $A_s \in \mathcal{F}_s$ that satisfies $\mathbb{P}[A_s] > 0$ and define $A_t := \emptyset$ for all $(s \neq) t \in \mathbb{N}_0$. Moreover, define $\tilde{\Upsilon}_t : \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta} \to \mathbb{R}$ by

$$\tilde{\Upsilon}_t(Y, \bar{X}) := \mathbb{E}[\Upsilon_t(Y, \bar{X}) I_{A_t}]$$

for each $t \in \mathbb{N}_0$. $\tilde{\Upsilon}_t$ is decreasing in the first and increasing in the second argument, convex and continuous for bounded decreasing sequences in the first argument and bounded increasing sequences in the second argument.

Consider the set $\mathcal{D}_{\tilde{\Upsilon}} := \{(Y, \bar{X}) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta} | 0 \geq \tilde{\Upsilon}_t(Y, \bar{X}) \text{ for all } t \in \mathbb{N}_0\}$. It follows analogously to the proof of Lemma 3.5 that for each $r > 0$ the set

$$\mathcal{C}_r := \mathcal{D}_{\tilde{\Upsilon}} \cap \{(Y, \bar{X}) \in L^\infty_{\mathcal{H}} \times (L^\infty_{\mathcal{H}})^n | \|(Y, \bar{X})\|_{(L^\infty_{\mathcal{H}})^{n+1}} \leq r\}$$

is closed in $(L^1_{\mathcal{H}})^{n+1}$. This implies again that $\mathcal{D}_{\tilde{\Upsilon}}$ is $\sigma((L^\infty_{\mathcal{H}})^{n+1}, (L^1_{\mathcal{H}})^{n+1})$-closed.

Similarly as in the proof of Lemma 3.5, we can conclude that this yields a contradiction to $\mathbb{P}[A_s] > 0$ and $0 < \Upsilon_s(Y^\#, \bar{X}^\#)$ on A_s, and hence $\mathcal{B}_\Lambda$ is $\sigma(\mathcal{R}^{\infty,n+1}, \mathcal{A}^{1,n+1})$-closed. □

Lemma 3.7. *Suppose that $\rho = \rho_0 \circ \Lambda$ is a conditional convex systemic risk measure characterized by a convex aggregation function Λ and a conditional convex single-firm risk measure ρ_0. If ρ_0 is continuous for bounded increasing sequences, then for each $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ the set*

$$\begin{aligned} \mathcal{C}^d_{\rho_0\circ\Lambda} := \{(V, X, Z, \bar{Z}) \in \mathcal{R}^{\infty n+3}_{\tau,\theta} | V = \phi^\# I_{[\tau,\infty)} \textit{ for some } \phi \in L^\infty_\tau \\ \textit{and } \mathbb{E}[\rho_0(\Lambda(\bar{X} - \bar{Z}) - X - Z)] - \mathbb{E}[\phi] \leq d\} \end{aligned}$$

is $\sigma(\mathcal{R}^{\infty,n+3}, \mathcal{A}^{1,n+3})$-closed for each $d \in \mathbb{R}$.

Proof. For $d \in \mathbb{R}$ define the map $\varrho : \mathcal{R}^{\infty,n+3}_{\tau,\theta} \to \mathbb{R}$, $\varrho(V,X,Z,\bar{Z}) := \mathbb{E}[\rho_0(\Lambda(\bar{X}-\bar{Z}) - X - Z)] - \mathbb{E}[V_\tau] - d$. Then for every decreasing sequence $(\tilde{U}^{(k)}) \subset \mathcal{R}^{\infty,n+3}_{\tau,\theta}$ with $\tilde{U}^{(k)}_t \downarrow \tilde{U}_t$ $\mathbb{P}$-a.s. for all $t \in \mathbb{N}_0$ and $\tilde{U} \in \mathcal{R}^{\infty,n+3}_{\tau,\theta}$, we obtain $\varrho(\tilde{U}^{(k)}) \uparrow \varrho(\tilde{U})$. Moreover, we have

$$\mathcal{C}^d_{\rho_0 \circ \Lambda} = \{(V,X,Z,\bar{Z}) \in \mathcal{R}^{\infty,n+3}_{\tau,\theta} | V = \phi I_{[\tau,\infty)} \text{ for } \phi \in L^\infty_\tau \text{ and } 0 \geq \varrho(\phi I_{[\tau,\infty)}, X, Z, \bar{Z})\}.$$

Again, for each $r > 0$, the set

$$\mathcal{C}_r := \mathcal{C}^d_{\rho_0 \circ \Lambda} \cap \{(V,X,Z,\bar{Z}) \in (L^\infty_{\mathcal{H}})^{n+3} | \|(V,X,Z,\bar{Z})\|_{(L^\infty_{\mathcal{H}})^{n+3}} \leq r\}$$

is closed in $(L^1_{\mathcal{H}})^{n+3}$. The assertion follows from the second part of Lemma 3.3. □

Because of the following lemma, we can define the topology $\sigma(\mathcal{R}^{\infty,m}_{\tau,\theta}, \mathcal{A}^{1,m}_{\tau,\theta})$ on $\mathcal{R}^{\infty,m}_{\tau,\theta}$ and the topology $\sigma(\mathcal{A}^{1,m}_{\tau,\theta}, \mathcal{R}^{\infty,m}_{\tau,\theta})$ on $\mathcal{A}^{1,m}_{\tau,\theta}$ analogously to $\sigma(\mathcal{R}^{\infty,m}, \mathcal{A}^{1,m})$ on $\mathcal{R}^{\infty,m}$ and $\sigma(\mathcal{A}^{1,m}, \mathcal{R}^{\infty,m})$ on $\mathcal{A}^{1,m}$.

Lemma 3.8. *$\mathcal{A}^{1,m}_{\tau,\theta}$ and $\mathcal{R}^{\infty,m}_{\tau,\theta}$ satisfy the following properties:*

1. *$\mathcal{R}^{\infty,m}_{\tau,\theta}$ separates points of $\mathcal{A}^{1,m}_{\tau,\theta}$ under $\langle \cdot,\cdot \rangle_m |_{\mathcal{R}^{\infty,m}_{\tau,\theta} \times \mathcal{A}^{1,m}_{\tau,\theta}}$: If $\bar{\xi} \in \mathcal{A}^{1,m}_{\tau,\theta}$ and $\langle \bar{X}, \bar{\xi} \rangle_m = 0$ for all $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$, then $\bar{\xi} = 0$.*
2. *$\mathcal{A}^{1,m}_{\tau,\theta}$ separates points of $\mathcal{R}^{\infty,m}_{\tau,\theta}$ under $\langle \cdot,\cdot \rangle_m |_{\mathcal{R}^{\infty,m}_{\tau,\theta} \times \mathcal{A}^{1,m}_{\tau,\theta}}$: If $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$ and $\langle \bar{X}, \bar{\xi} \rangle_m = 0$ for all $\bar{\xi} \in \mathcal{A}^{1,m}_{\tau,\theta}$, then $\bar{X} = 0$.*

Proof. Note that $\mathcal{R}^{\infty,m}$ separates points of $\mathcal{A}^{1,m}$ and $\mathcal{A}^{1,m}$ separates points of $\mathcal{R}^{\infty,m}$ under $\langle \cdot,\cdot \rangle_m$. Fix $\bar{\xi} \in \mathcal{A}^{1,m}_{\tau,\theta}$ and suppose that $\langle \bar{X}, \bar{\xi} \rangle_m = 0$ for all $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$. For each $\bar{Y} \in \mathcal{R}^\infty$, we obtain $\langle \bar{Y}, \bar{\xi} \rangle_m = \langle \bar{Z}, \bar{\xi} \rangle_m$ for $\bar{Z} := \bar{Y} I_{[\tau,\theta]} + \bar{Y}_\theta I_{(\theta,\infty)} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$. This implies that $\langle \bar{Y}, \bar{\xi} \rangle_m = 0$ for all $\bar{Y} \in \mathcal{R}^\infty$, and thus $\bar{\xi} = 0$.

To show the second assertion, consider $\bar{X} \in \mathcal{R}^{\infty,m}_{\tau,\theta}$ and suppose that $\langle \bar{X}, \bar{\phi} \rangle_m = 0$ for all $\bar{\phi} \in \mathcal{A}^{1,m}_{\tau,\theta}$. For every $\bar{\xi} \in \mathcal{A}^{1,m}$, we have $\langle \bar{X}, \bar{\xi} \rangle_m = \sum_{i=1}^m \mathbb{E}[\sum_{t \in \mathbb{N}_0} \bar{X}^i_t \Delta \xi_t]$ and there exists a $\bar{\phi} \in \mathcal{A}^{1,m}_{\tau,\theta}$ with $\langle \bar{X}, \bar{\xi} \rangle_m = \langle \bar{X}, \bar{\phi} \rangle_m$. Therefore, $\langle \bar{X}, \bar{\psi} \rangle_m = 0$ for all $\bar{\psi} \in \mathcal{A}^{1,m}$, which implies that $\bar{X} = 0$. □

With the techniques from the previous proof we can easily show that the topology

$$\mathfrak{T}^m_{\tau,\theta} := \{\mathcal{U} \cap \mathcal{R}^{\infty,m}_{\tau,\theta} | \mathcal{U} \in \sigma(\mathcal{R}^{\infty,m}, \mathcal{A}^{1,m})\}.$$

on $\mathcal{R}^\infty_{\tau,\theta}$ is compatible with the pairing $\langle \cdot,\cdot \rangle_m |_{\mathcal{R}^{\infty,m}_{\tau,\theta} \times \mathcal{A}^{1,m}_{\tau,\theta}}$. Therefore, we finally obtain the following result.

Lemma 3.9. *Let ρ_0 be a conditional convex single-firm risk measure and Λ be a convex aggregation function.*

1. *If ρ_0 is continuous for bounded increasing sequences, then $\mathcal{B}_{\rho_0}$ is $\sigma(\mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta}, \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^1_{\tau,\theta})$-closed.*
2. *$\mathcal{B}_\Lambda$ is $\sigma(\mathcal{R}^{\infty,n+1}_{\tau,\theta}, \mathcal{A}^{1,n+1}_{\tau,\theta})$-closed.*

3. *If ρ_0 is continuous for bounded increasing sequences, then $\mathcal{C}^d_{\rho_0\circ\Lambda}$ is $\sigma(\mathcal{R}^{\infty,n+3}_{\tau,\theta},\mathcal{A}^{1,n+3}_{\tau,\theta})$-closed for all $d\in\mathbb{R}$.*

Proof. Since every set $\mathcal{C}\subset\mathcal{R}^{\infty,m}_{\tau,\theta}$ which is $\mathfrak{T}^m_{\tau,\theta}$-closed satisfies $\mathcal{C}=\mathcal{D}\cap\mathcal{R}^{\infty,m}_{\tau,\theta}$ for some $\sigma(\mathcal{R}^{\infty,m},\mathcal{A}^{1,m})$-closed set $\mathcal{D}\subset\mathcal{R}^{\infty,m}$, the previous results (Lemma 3.5 - Lemma 3.7) yield that $\mathcal{B}_{\rho_0}$ is $\mathfrak{T}^2_{\tau,\theta}$-closed, $\mathcal{B}_\Lambda$ is $\mathfrak{T}^{n+1}_{\tau,\theta}$-closed and $\mathcal{C}^d_{\rho_0\circ\Lambda}$ is $\mathfrak{T}^{n+3}_{\tau,\theta}$-closed for all $d\in\mathbb{R}$. The assertion follows from Theorem 5.98 in Aliprantis and Border (2006). □

3.3 Dual representation

Let us first consider some generalizations of the concepts in Cheridito et al. (2006) for m dimensions. For $\bar{X}\in\mathcal{R}^{\infty,m}$ and $\bar{\xi}\in\mathcal{A}^{1,m}$, set

$$\langle\bar{X},\bar{\xi}\rangle^{\tau,\theta}_m := \sum_{i=1}^m \mathbb{E}\left[\sum_{t\in[\tau,\theta]\cap\mathbb{N}_0}\bar{X}^i_t\Delta\bar{\xi}^i_t\,\middle|\,\mathcal{F}_\tau\right]$$

and $\langle\cdot,\cdot\rangle^{\tau,\theta}:=\langle\cdot,\cdot\rangle^{\tau,\theta}_1$. Then for $\bar{\xi}\in\mathcal{A}^{1,m}_{\tau,\theta}$ and $\bar{X}\in\mathcal{R}^{\infty,m}$, we have the following equations:

$$\langle\bar{X},\bar{\xi}\rangle_m=\sum_{i=1}^m\langle\bar{X}^i,\bar{\xi}^i\rangle=\sum_{i=1}^m\mathbb{E}[\langle\bar{X}^i,\bar{\xi}^i\rangle^{\tau,\theta}]=\mathbb{E}[\langle\bar{X},\bar{\xi}\rangle^{\tau,\theta}_m].$$

Moreover, define

$$\mathcal{A}^{1,m}_+ := \{\bar{\xi}\in\mathcal{A}^{1,m}\,|\,\Delta\bar{\xi}^i_t\geq 0 \text{ for all } t\in\mathbb{N}_0, i=1,\dots,m\},\quad (\mathcal{A}^{1,m}_{\tau,\theta})_+ := p^{\tau,\theta}_m\mathcal{A}^{1,m}_+$$

(with $\mathcal{A}^1_+:=\mathcal{A}^{1,1}_+$ and $(\mathcal{A}^1_{\tau,\theta})_+:=(\mathcal{A}^{1,1}_{\tau,\theta})_+$) and

$$\mathcal{D}_{\tau,\theta}:=\{\xi\in(\mathcal{A}^1_{\tau,\theta})_+\,|\,\langle 1,\xi\rangle^{\tau,\theta}=1\}\quad\text{and}\quad\mathcal{E}_{\tau,\theta}:=\{\xi\in(\mathcal{A}^1_{\tau,\theta})_+\,|\,\langle 1,\xi\rangle^{\tau,\theta}\leq 1\}.$$

Finally, let $L^0_\tau(\overline{\mathbb{R}}_+)$ the space of extended random variables γ that are $\mathcal{F}_\tau$-measurable and satisfy $\gamma\geq 0$.

Theorem 3.10. *Suppose that $\rho=\rho_0\circ\Lambda$ is a conditional convex systemic risk measure characterized by a convex aggregation function Λ and a conditional convex single-firm risk measure ρ_0. If ρ_0 is continuous for bounded increasing sequences, then ρ admits for any $\bar{X}\in\mathcal{R}^{\infty,n}_{\tau,\theta}$ the representation*

$$\rho(\bar{X})=\operatorname*{ess\,sup}_{(\xi,\bar{\xi})\in\mathcal{E}_{\tau,\theta}\times(\mathcal{A}^{1,n}_{\tau,\theta})_+}\{\langle\bar{X},\bar{\xi}\rangle^{\tau,\theta}_n-\alpha^{\tau,\theta}_n(\xi,\bar{\xi})\}\tag{3}$$

with $\alpha^{\tau,\theta}_n:\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}\to L^0_\tau(\overline{\mathbb{R}}_+)$ that is given by

$$\alpha^{\tau,\theta}_n(\xi,\bar{\xi})=\operatorname*{ess\,sup}_{(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0},(V,\bar{Z})\in\mathcal{B}_\Lambda}\{-\gamma+\langle Y-V,\xi\rangle^{\tau,\theta}+\langle\bar{Z},\bar{\xi}\rangle^{\tau,\theta}_n\}.\tag{4}$$

If additionally $\rho(\mathbb{R}^nI_{[\tau,\infty)})=\mathbb{R}$, then the first component of a solution to optimization problem (3) satisfies $\xi\in\mathcal{D}_{\tau,\theta}$.

Remark. Note that compared to the corresponding theorem in Kromer et al. (2014) we do not require Λ to have an additional continuity property.

Proof. Part 1: We will prove that

$$\mathbb{E}[\rho(\bar{X})] = \mathbb{E}\left[\operatorname*{ess\,sup}_{(\xi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}} \{\langle \bar{X}, \bar{\xi}\rangle_n^{\tau,\theta} - \alpha_n^{\tau,\theta}(\xi,\bar{\xi})\}\right].$$

By Proposition 3.2, ρ satisfies

$$\rho(\bar{X}) = \operatorname*{ess\,inf}_{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta}} \{\gamma + \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) + \iota_{\mathcal{B}_\Lambda}(Y,\bar{X})\}$$

where $\iota_{\mathcal{D}} : \mathcal{E} \to \mathbb{R}\cup\{+\infty\}$, $\mathcal{D}\subset\mathcal{E}\in\{\mathcal{R}^\infty_{\tau,\theta}\times\mathcal{R}^\infty_{\tau,\theta}, \mathcal{R}^\infty_{\tau,\theta}\times\mathcal{R}^{\infty,n}_{\tau,\theta}\}$, is given by $\iota_{\mathcal{D}}(x) := 0$ if $x\in\mathcal{D}$ and $\iota_{\mathcal{D}}(x) := +\infty$ if $x\notin\mathcal{D}$. Consider $s_{\mathcal{B}_{\rho_0}} : \mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta}\to L^0_\tau(\overline{\mathbb{R}}_+)$ for the convex set $\mathcal{B}_{\rho_0}$ defined by

$$s_{\mathcal{B}_{\rho_0}}(-\psi,\xi) := \operatorname*{ess\,sup}_{(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0}} \{-\langle \gamma I_{[\tau,\infty)}, \psi\rangle^{\tau,\theta} + \langle Y,\xi\rangle^{\tau,\theta}\}.$$

Then one can easily show that $\{-\langle \gamma I_{[\tau,\infty)}, \psi\rangle^{\tau,\theta} + \langle Y,\xi\rangle^{\tau,\theta} | (\gamma I_{[\tau,\infty)}, Y)\in\mathcal{B}_{\rho_0}\}$ is directed upwards. Hence, the function $\tilde{s}_{\mathcal{B}_{\rho_0}} : \mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta}\to\mathbb{R}\cup\{+\infty\}$ defined by $\tilde{s}_{\mathcal{B}_{\rho_0}}(\psi,\xi) := \mathbb{E}[s_{\mathcal{B}_{\rho_0}}(\psi,\xi)]$ satisfies

$$\tilde{s}_{\mathcal{B}_{\rho_0}}(-\psi,\xi) = \sup_{(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0}} \{-\langle \gamma I_{[\tau,\infty)}, \psi\rangle + \langle Y,\xi\rangle\} = \iota^*_{\mathcal{B}_{\rho_0}}(-\psi,\xi) \quad \text{for } (\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta}$$

where $\iota^*_{\mathcal{B}_{\rho_0}}$ denotes the convex conjugate of $\iota_{\mathcal{B}_{\rho_0}}$. By Lemma 3.9, $\mathcal{B}_{\rho_0}$ is $\sigma(\mathcal{R}^\infty_{\tau,\theta}\times\mathcal{R}^\infty_{\tau,\theta}, \mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta})$-closed, and therefore $\iota_{\mathcal{B}_{\rho_0}}$ is also $\sigma(\mathcal{R}^\infty_{\tau,\theta}\times\mathcal{R}^\infty_{\tau,\theta}, \mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta})$-closed.

The duality theorem for conjugate functions (see for instance Theorem 5 in Rockafellar (1974)) yields

$$\begin{aligned}\iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) = \iota^{**}_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) &= \sup_{(\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta}} \{\langle \gamma I_{[\tau,\infty)}, \psi\rangle + \langle Y,\xi\rangle - \iota^*_{\mathcal{B}_{\rho_0}}(\psi,\xi)\} \\ = \sup_{(\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta}} &\mathbb{E}[-\langle \gamma I_{[\tau,\infty)}, \psi\rangle^{\tau,\theta} + \langle Y,\xi\rangle^{\tau,\theta} - s_{\mathcal{B}_{\rho_0}}(-\psi,\xi)] \end{aligned} \tag{5}$$

for $(\gamma, Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta}$. Similarly, we can define $s_{\mathcal{B}_\Lambda} : \mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}\to L^0_\tau(\overline{\mathbb{R}}_+)$ by

$$s_{\mathcal{B}_\Lambda}(-\phi,\bar{\xi}) := \operatorname*{ess\,sup}_{(Y,\bar{Z})\in\mathcal{B}_\Lambda} \{-\langle Y,\phi\rangle^{\tau,\theta} + \langle \bar{Z},\bar{\xi}\rangle_n^{\tau,\theta}\}$$

and show that $\tilde{s}_{\mathcal{B}_\Lambda} : \mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}\to\mathbb{R}\cup\{+\infty\}$ defined by $\tilde{s}_{\mathcal{B}_\Lambda}(\phi,\bar{\xi}) := \mathbb{E}[s_{\mathcal{B}_\Lambda}(\phi,\bar{\xi})]$ satisfies $\tilde{s}_{\mathcal{B}_\Lambda}(-\phi,\bar{\xi}) = \sup_{(Y,\bar{Z})\in\mathcal{B}_\Lambda}\{-\langle Y,\phi\rangle + \langle \bar{Z},\bar{\xi}\rangle_n\} = \iota^*_{\mathcal{B}_\Lambda}(-\phi,\bar{\xi})$ for all $(\phi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}$. Moreover, by Lemma 3.9, $\mathcal{B}_\Lambda$ is $\sigma(\mathcal{R}^{\infty,n+1}_{\tau,\theta}, \mathcal{A}^{1,n+1}_{\tau,\theta})$-closed, and therefore $\iota_{\mathcal{B}_\Lambda}$ is a $\sigma(\mathcal{R}^{\infty,n+1}_{\tau,\theta}, \mathcal{A}^{1,n+1}_{\tau,\theta})$-closed function. It follows that

$$\begin{aligned}\iota_{\mathcal{B}_\Lambda}(Y,\bar{X}) = \iota^{**}_{\mathcal{B}_\Lambda}(Y,\bar{X}) &= \sup_{(\phi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}} \{\langle Y,\phi\rangle + \langle \bar{X},\bar{\xi}\rangle_n - \iota^*_{\mathcal{B}_\Lambda}(\phi,\bar{\xi})\} \\ = \sup_{(\phi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}} &\mathbb{E}[-\langle Y,\phi\rangle^{\tau,\theta} + \langle \bar{X},\bar{\xi}\rangle_n^{\tau,\theta} - s_{\mathcal{B}_\Lambda}(-\phi,\bar{\xi})]\end{aligned} \tag{6}$$

for $(Y,\bar{X}) \in \mathcal{R}^{\infty}_{\tau,\theta} \times \mathcal{R}^{\infty;n}_{\tau,\theta}$. Since $\{\gamma + \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) + \iota_{\mathcal{B}_\Lambda}(Y,\bar{X}) \,|\, (\gamma, Y) \in L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta}\}$ is directed downwards, we obtain with Equations (5) and (6) that

$$\begin{aligned}
\mathbb{E}[\rho(\bar{X})] &= \mathbb{E}\Bigg[\operatorname*{ess\,inf}_{(\gamma,Y)\in L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta}} \{\gamma + \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) + \iota_{\mathcal{B}_\Lambda}(Y,\bar{X})\Bigg] \\
&= \inf_{(\gamma,Y)\in L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta}} \sup_{\substack{(\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta},\\ (\phi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}}} \Bigg\{\mathbb{E}\left[\gamma\right] - \langle \gamma I_{[\tau,\infty)}, \psi\rangle + \langle Y, \xi\rangle - \langle Y, \phi\rangle \\
&\qquad + \langle \bar{X}, \bar{\xi}\rangle_n - \tilde{s}_{\mathcal{B}_{\rho_0}}(-\psi,\xi) - \tilde{s}_{\mathcal{B}_\Lambda}(-\phi,\bar{\xi})]\Bigg\}
\end{aligned} \tag{7}$$

By Theorem 6 and 7 in Rockafellar (1974) and the third part in Lemma 3.9, we can exchange the infimum and the supremum in (7). This means

$$\begin{aligned}
\mathbb{E}[\rho(\bar{X})] &= \sup_{\substack{(\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta},\\ (\phi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}}} \inf_{(\gamma,Y)\in L^\infty_\tau \times \mathcal{R}^\infty_{\tau,\theta}} \Bigg\{\mathbb{E}\Bigg[\gamma\Big(1 - \sum_{t\in\mathbb{N}_0} \Delta\psi_t\Big)\Bigg] + \langle Y, \xi - \phi\rangle \\
&\qquad + \mathbb{E}\left[\langle \bar{X}, \bar{\xi}\rangle^{\tau,\theta}_n - s_{\mathcal{B}_{\rho_0}}(-\psi,\xi) - s_{\mathcal{B}_\Lambda}(-\phi,\bar{\xi})\right]\Bigg\} \\
&= \sup_{\substack{(\psi,\xi)\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^1_{\tau,\theta},\langle 1,\psi\rangle^{\tau,\theta}=1\\ (\xi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}}} \Bigg\{\mathbb{E}\left[\langle \bar{X}, \bar{\xi}\rangle^{\tau,\theta}_n - s_{\mathcal{B}_{\rho_0}}(-\psi,\xi) - s_{\mathcal{B}_\Lambda}(-\xi,\bar{\xi})\right]\Bigg\}.
\end{aligned}$$

Moreover, note that $(\xi,\bar{\xi}) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta}$ and $\psi \in \mathcal{A}^1_{\tau,\theta}$ with $\langle 1, \psi\rangle^{\tau,\theta} = 1$ we have

$$s_{\mathcal{B}_{\rho_0}}(-\psi,\xi) + s_{\mathcal{B}_\Lambda}(-\xi,\bar{\xi}) = \operatorname*{ess\,sup}_{(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0},(V,\bar{Z})\in\mathcal{B}_\Lambda} \{-\gamma + \langle Y - V, \xi\rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi}\rangle^{\tau,\theta}_n\} = \alpha^{\tau,\theta}_n.$$

Since $\{\langle \bar{X}, \bar{\xi}\rangle^{\tau,\theta}_n - \alpha^{\tau,\theta}_n(\xi,\bar{\xi}) | (\xi,\bar{\xi}) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta}\}$ is directed upwards, it follows

$$\begin{aligned}
\mathbb{E}[\rho(\bar{X})] &= \sup_{(\xi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}} \{\mathbb{E}[\langle \bar{X}, \bar{\xi}\rangle^{\tau,\theta}_n - \alpha^{\tau,\theta}_n(\xi,\bar{\xi})]\} \\
&= \mathbb{E}\Bigg[\operatorname*{ess\,sup}_{(\xi,\bar{\xi})\in\mathcal{A}^1_{\tau,\theta}\times\mathcal{A}^{1,n}_{\tau,\theta}} \{\langle \bar{X}, \bar{\xi}\rangle^{\tau,\theta}_n - \alpha^{\tau,\theta}_n(\xi,\bar{\xi})\}\Bigg].
\end{aligned}$$

Part 2: In this part of the proof we will show that

$$\mathbb{E}[\rho(\bar{X})] = \sup_{\mathcal{E}_{\tau,\theta}\times(\mathcal{A}^{1,n}_{\tau,\theta})_+} \{\mathbb{E}[\langle \bar{X}, \bar{\xi}\rangle^{\tau,\theta}_n - \alpha^{\tau,\theta}_n(\xi,\bar{\xi})]\} = \mathbb{E}\Bigg[\operatorname*{ess\,sup}_{(\xi,\bar{\xi})\in\mathcal{E}_{\tau,\theta}\times(\mathcal{A}^{1,n}_{\tau,\theta})_+} \{\langle \bar{X}, \bar{\xi}\rangle^{\tau,\theta}_n - \alpha^{\tau,\theta}_n(\xi,\bar{\xi})\}\Bigg].$$

For $(\xi,\bar{\xi}) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^1_{\tau,\theta}$ the following equality holds:

$$\begin{aligned}
\mathbb{E}[\alpha^{\tau,\theta}_n(\xi,\bar{\xi})] &= \mathbb{E}\Bigg[\operatorname*{ess\,sup}_{(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0},(V,\bar{Z})\in\mathcal{B}_\Lambda} \{-\gamma + \langle Y - V, \xi\rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi}\rangle^{\tau,\theta}_n\}\Bigg] \\
&= \sup_{(\gamma I_{[\tau,\infty)},Y)\in\mathcal{B}_{\rho_0},(V,\bar{Z})\in\mathcal{B}_\Lambda} \{-\mathbb{E}[\gamma] + \langle Y - V, \xi\rangle + \langle \bar{Z}, \bar{\xi}\rangle_n\}.
\end{aligned}$$

Now, suppose that there exists $s \in \mathbb{N}_0$ such that $\Delta\xi_s < 0$ on $A_s \in \mathcal{F}_s$ such that $\mathbb{P}[A_s] > 0$. Define $Y^{(m)} \in \mathcal{R}^\infty_{\tau,\theta}$ by $Y^{(m)}_t := -mI_{A_s}I_{\{t=s\}}$ for $t \in \mathbb{N}_0$. Since $0 \geq Y^{(m)}$ for all $m \in \mathbb{N}$, we obtain

$$\langle Y^{(m)}, \xi\rangle = \mathbb{E}\left[\sum_{t\in\mathbb{N}_0} Y^{(m)}_t \Delta\xi_t\right] = \mathbb{E}\left[-mI_{A_s}\Delta\xi_s\right] > 0.$$

Moreover, since $(0,0) \in \mathcal{B}_{\rho_0}$, the monotonicity of $\mathcal{B}_{\rho_0}$ implies $(0, Y^{(m)}) \in \mathcal{B}_{\rho_0}$ for each $m \in \mathbb{N}$. Because $\langle Y^{(m)}, \xi\rangle$ increases to ∞ for $m \to \infty$, it follows $\mathbb{E}[\alpha^{\tau,\theta}_n(\xi,\bar{\xi})] = \infty$. This means that it suffices to consider $\xi \in (\mathcal{A}^1_{\tau,\theta})_+$. Similarly, we obtain the remaining assertions for feasible solutions.

Part 3: Finally, we will verify Equation (3). First, we will show the inequality " $\geq$ ": For $(\xi,\bar{\xi}) \in \mathcal{E}_{\tau,\theta} \times (\mathcal{A}^{1,n}_{\tau,\theta})_+$ we have

$$\begin{aligned}
&\alpha^{\tau,\theta}_n(\xi,\bar{\xi}) \\
&= \operatorname*{ess\,sup}_{\substack{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta},\\ (V,\bar{Z})\in\mathcal{R}^\infty_{\tau,\theta}\times\mathcal{R}^{\infty,n}_{\tau,\theta}}} \left\{-\gamma + \langle Y - V, \xi\rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi}\rangle^{\tau,\theta}_n - \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) - \iota_{\mathcal{B}_\Lambda}(V,\bar{Z})\right\} \\
&\geq \operatorname*{ess\,sup}_{(\gamma,Y)\in L^\infty_\tau\times\mathcal{R}^\infty_{\tau,\theta}, \bar{Z}\in\mathcal{R}^{\infty,n}_{\tau,\theta}} \left\{-\gamma + \langle \bar{Z}, \bar{\xi}\rangle^{\tau,\theta}_n - \iota_{\mathcal{B}_{\rho_0}}(\gamma I_{[\tau,\infty)}, Y) - \iota_{\mathcal{B}_\Lambda}(Y,\bar{Z})\right\} \\
&= \operatorname*{ess\,sup}_{\bar{Z}\in\mathcal{R}^{\infty,n}_{\tau,\theta}} \left\{\langle \bar{Z}, \bar{\xi}\rangle^{\tau,\theta}_n - \rho(\bar{Z})\right\}
\end{aligned} \tag{8}$$

where the last equality follows from the primal representation. It follows from (8) that $\rho(\bar{Z}) \geq \langle \bar{Z}, \bar{\xi}\rangle^{\tau,\theta}_n - \alpha^{\tau,\theta}_n(\xi,\bar{\xi})$ for all $\bar{Z} \in \mathcal{R}^\infty_{\tau,\theta}$ and $(\xi,\bar{\xi}) \in \mathcal{E}_{\tau,\theta} \times (\mathcal{A}^{1,n}_{\tau,\theta})_+$, which implies that

$$\rho(\bar{Z}) \geq \operatorname*{ess\,sup}_{(\xi,\bar{\xi})\in\mathcal{E}_{\tau,\theta}\times(\mathcal{A}^{1,n}_{\tau,\theta})_+} \left\{\langle \bar{Z}, \bar{\xi}\rangle^{\tau,\theta}_n - \alpha^{\tau,\theta}_n(\xi,\bar{\xi})\right\} \quad \text{for all } \bar{Z} \in \mathcal{R}^\infty_{\tau,\theta}. \tag{9}$$

Finally, the assertion follows together with the result from part 2. □

It remains to consider the positively homogeneous special case of Theorem 3.10. Define

$$\mathcal{Z} := \left\{(\gamma I_{[\tau,\infty)}, Y, V, \bar{Z}) \in \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^\infty_{\tau,\theta} \times \mathcal{R}^{\infty,n}_{\tau,\theta} \,|\, (\gamma I_{[\tau,\infty)}, Y) \in \mathcal{B}_{\rho_0}, (V,\bar{Z}) \in \mathcal{B}_\Lambda\right\}$$

and

$$\begin{aligned}
\mathcal{Z}^\# := \{&(\xi,\bar{\xi}) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta} \,|\, -\gamma + \langle Y - V, \xi\rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi}\rangle^{\tau,\theta}_n \leq 0 \\
&\text{for all } (\gamma I_{[\tau,\infty)}, Y, V, \bar{Z}) \in \mathcal{Z}\}.
\end{aligned}$$

Theorem 3.11. *Suppose that $\rho = \rho_0 \circ \Lambda$ is a conditional positively homogeneous systemic risk measure characterized by a positively homogeneous aggregation function Λ and a conditional positively homogeneous single-firm risk measure ρ_0. If ρ_0 is continuous for bounded increasing sequences, then ρ admits for any $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ the representation*

$$\rho(\bar{X}) = \operatorname*{ess\,sup}_{(\xi,\bar{\xi})\in\mathcal{Z}^\#} \langle \bar{X}, \bar{\xi}\rangle^{\tau,\theta}_n. \tag{10}$$

In addition, a solution to optimization problem (10) *must satisfy*

$$\xi \in \mathcal{E}_{\tau,\theta}, \quad \bar{\xi} \in (\mathcal{A}^{1,n}_{\tau,\theta})_+ \quad \textit{and} \quad \langle 1_n, \bar{\xi} \rangle_n^{\tau,\theta} \leq n \langle 1, \xi \rangle^{\tau,\theta}.$$

If additionally $\rho(\mathbb{R}^n I_{[\tau,\infty)}) = \mathbb{R}$*, then* $\xi \in \mathcal{D}_{\tau,\theta}$*.*

Proof. By Theorem 3.10, ρ admits for all $\bar{X} \in \mathcal{R}^{\infty,n}_{\tau,\theta}$ the representation

$$\rho(\bar{X}) = \operatorname*{ess\,sup}_{(\xi,\bar{\xi}) \in \mathcal{E}^1_{\tau,\theta} \times (\mathcal{A}^{1,n}_{\tau,\theta})_+} \{ \langle \bar{X}, \bar{\xi} \rangle_n^{\tau,\theta} - \alpha_n^{\tau,\theta}(\xi, \bar{\xi}) \} \tag{11}$$

with $\alpha_n^{\tau,\theta} : \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta} \to L_\tau(\overline{\mathbb{R}}_+)$ defined by

$$\alpha_n^{\tau,\theta}(\xi, \bar{\xi}) = \operatorname*{ess\,sup}_{(\gamma I_{[\tau,\infty)}, Y, V, \bar{Z}) \in \mathcal{Z}} \{ -\gamma + \langle Y - V, \xi \rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi} \rangle_n^{\tau,\theta} \}.$$

Since $\mathcal{B}_{\rho_0}$ and $\mathcal{B}_\Lambda$ are $\mathcal{F}_\tau$-cones, one can easily verify (similarly as in Corollary 11.6 in Föllmer and Schied (2011)) that

$$\{\alpha_n^{\tau,\theta}(\xi, \bar{\xi}) = 0\} \cup \{\alpha_n^{\tau,\theta}(\xi, \bar{\xi}) = \infty\} = \Omega \quad \text{for all } (\xi, \bar{\xi}) \in \mathcal{A}^1_{\tau,\theta} \times \mathcal{A}^{1,n}_{\tau,\theta}.$$

The definition of $\mathcal{Z}^{\#}$ implies $-\gamma + \langle Y - V, \xi \rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi} \rangle_n^{\tau,\theta} \leq 0$ for all $(\gamma I_{[\tau,\infty)}, Y) \in \mathcal{B}_{\rho_0}$ and $(V, \bar{Z}) \in \mathcal{B}_\Lambda$. It follows that $\alpha_n^{\tau,\theta}(\xi, \bar{\xi}) = 0$ for all $(\xi, \bar{\xi}) \in \mathcal{Z}^{\#}$.

If $(\xi, \bar{\xi}) \notin \mathcal{Z}^{\#}$ we can find an element $(\gamma I_{[\tau,\infty)}, Y, V, \bar{Z}) \in \mathcal{Z}$ and $B \in \mathcal{F}_\tau$ with $\mathbb{P}[B] > 0$ such that

$$(-\gamma + \langle Y - V, \xi \rangle^{\tau,\theta} + \langle \bar{Z}, \bar{\xi} \rangle_n^{\tau,\theta}) I_B = -\gamma I_B + \langle Y I_B - V I_B, \xi \rangle^{\tau,\theta} + \langle \bar{Z} I_B, \bar{\xi} \rangle_n^{\tau,\theta} > 0 \quad \text{on } B.$$

Since $(\lambda \gamma I_B I_{[\tau,\infty)}, \lambda Y I_B, \lambda V I_B, \lambda \bar{Z} I_B) \in \mathcal{Z}$ for all $\lambda > 0$ and $\lim_{\lambda \to \infty} \lambda(-\gamma I_B + \langle Y I_B - V I_B, \xi \rangle^{\tau,\theta} + \langle \bar{Z} I_B, \bar{\xi} \rangle_n^{\tau,\theta}) = \infty I_B$ $\mathbb{P}$-a.s., we obtain $\alpha_n^{\tau,\theta}(\xi, \bar{\xi}) = \infty$ on B. Therefore, it suffices to consider $(\xi, \bar{\xi}) \in \mathcal{Z}^{\#}$ in (11).

The last property for feasible solutions to optimization problem (11) follows from $(0,0) \in \mathcal{B}_{\rho_0}$ and $(f_\Lambda(1) I_{[\tau,\infty)}, 1_n I_{[\tau,\infty)}) \in \mathcal{B}_\Lambda$ together with $f_\Lambda(1) = \Lambda(1_n) = n$. □

4 Dynamic systemic risk measures

Our next aim is to define dynamic systemic risk measures by using conditional systemic risk measures and analyze time-consistency properties. For the remaining part of this section fix $S \in \mathbb{N}_0$ and $T \in \mathbb{N}_0$ such that $S \leq T$ and define $\mathcal{S} := [S, T] \cap \mathbb{N}_0$. Note that as in the approach in Cheridito and Kupper (2011) we exclude the case $T = +\infty$.

Definition 4.1. *For* $t \in \mathcal{S}$*, consider a conditional convex [positively homogeneous] systemic risk measure* $\rho_{t,T} : \mathcal{R}^{\infty,n}_{t,T} \to L^\infty_t$ *with* $\rho_{t,T} = \rho^0_{t,T} \circ \Lambda_{t,T}$ *for a conditional convex [positively homogeneous] single-firm risk measure* $\rho^0_{t,T} : \mathcal{R}^\infty_{t,T} \to L^\infty_t$ *that satisfies the* $\mathcal{F}_t$*-translation property and a convex [positively homogeneous] aggregation function* $\Lambda_{t,T} : \mathbb{R}^n \to \mathbb{R}$*. We call the family* $(\rho_{t,T})_{t \in \mathcal{S}}$ dynamic convex [positively homogeneous] systemic risk measure*. Moreover, we call* $(\rho^0_{t,T})_{t \in \mathcal{S}}$ dynamic convex [positively homogeneous] single-firm risk measure *and* $(\Lambda_{t,T})_{t \in \mathcal{S}}$ dynamic convex [positively homogeneous] aggregation function.

In case of $\operatorname{Im}\Lambda_{t,T} = \mathbb{R}_+$ for each $t \in \mathcal{S}$, the corresponding function $f_{\Lambda_{t,T}}$ in the $f_{\Lambda_{t,T}}$-constancy property is a map from $\mathbb{R}$ to $\mathbb{R}_+$. Moreover, there exists $b_{t,T} \in \mathbb{R}_+$ such that $f_{\Lambda_{t,T}}|_{[b_{t,T},\infty)}$ is a bijective increasing function from $[b_{t,T},\infty)$ to $\mathbb{R}_+$ with $f_{\Lambda_{t,T}}(a) = 0$ for all $a \leq b_{t,T}$.

Note that we use only conditional single-firm risk measures $\rho^0_{t,T}$ that satisfy the $\mathcal{F}_t$-translation property. Since standard conditional convex risk measures from Cheridito et al. (2006) admit this property by definition, this is a feasible assumption. Nevertheless, $\rho_{t,T}$ does not satisfy a translation property because we do not assume that $\Lambda_{t,T}$ also satisfies some sort of translation property. Furthermore, one can easily show that in case of single-firm risk measures that satisfy the $\mathcal{F}_t$-translation property, the following property is satisfied:

(s7) $\rho_{t,T}(\gamma 1_n I_{[t,\infty)}) = f_{\rho_{t,T}}(\gamma)$ for each $t \in \mathcal{S}$ and $\gamma \in L^\infty_t$.

4.1 Time-consistency

For the dynamization of conditional systemic risk measures we need a specific time consistency property which establishes a connection between individual conditional systemic risk measures. Many approaches that consider conditional risk measures on stochastic processes study strong time-consistency. In the same line, we also use a variant of strong time-consistency for dynamic convex systemic risk measures. In case of risk measurement for only one firm, we use the concepts from Cheridito et al. (2006) and Cheridito and Kupper (2011).

Definition 4.2. *Let $(\rho^0_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex single-firm risk measure. Then $(\rho^0_{t,T})_{t\in\mathcal{S}}$ is* time-consistent *if the following property is satisfied:*

(r-TC) *For for every pair $s, t \in \mathcal{S}$ with $s \leq t$ and $X, Y \in \mathcal{R}^\infty_{s,T}$,*

$$XI_{[s,t)} = YI_{[s,t)} \text{ and } \rho^0_{t,T}(X) \leq \rho^0_{t,T}(Y)$$

imply $\rho^0_{s,T}(X) \leq \rho^0_{s,T}(Y)$.

Time-consistent dynamic convex aggregation functions are defined as follows.

Definition 4.3. $(\Lambda_{t,T})_{t\in\mathcal{S}}$ *is a* time-consistent dynamic convex aggregation function *if the following two properties are satisfied:*

(a-TC1) *Either all $\Lambda_{t,T}$ map to $\mathbb{R}$ or all $\Lambda_{t,T}$ map to $\mathbb{R}_+$.*

(a-TC2) *For for every pair $s, t \in \mathcal{S}$ with $s \leq t$ and $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$,*

$$\Lambda_{t,T}(\bar{X}I_{[t,\infty)}) \leq \Lambda_{t,T}(\bar{Y}I_{[t,\infty)})$$

implies $\Lambda_{s,T}(\bar{X}I_{[t,\infty)}) \leq \Lambda_{s,T}(\bar{Y}I_{[t,\infty)})$.

In property (a-TC2) we consider two loss processes after time t. If the aggregated loss process of one economy is greater than the aggregated loss process of another economy at time t, then this inequality is still satisfied if we consider the aggregation at time s.

Note that the following property and (a-TC2) are equivalent.

(a-TC2') For every pair $s, t \in \mathcal{S}$ with $s \leq t$ and $\bar{x}, \bar{y} \in \mathbb{R}^n$,

$$\Lambda_{t,T}(\bar{x}) \leq \Lambda_{t,T}(\bar{y})$$

implies $\Lambda_{s,T}(\bar{x}) \leq \Lambda_{s,T}(\bar{y})$.

If $(\Lambda_{t,T})_{t \in \mathcal{S}}$ is a time-consistent convex aggregation, then it follows from $\Lambda_{t,T}(\bar{X} I_{[t,\infty)}) = 0$ for $\bar{X} \in \mathcal{R}^{\infty,n}_{s,T}$ that $\Lambda_{s,T}(\bar{X} I_{[t,\infty)}) = 0$.

Definition 4.2 and Definition 4.3 are the basis to define time-consistent dynamic convex risk measures:

Definition 4.4. *Let $(\rho_{t,T})_{t \in \mathcal{S}}$ be a dynamic convex systemic risk measure. Then $(\rho_{t,T})_{t \in \mathcal{S}}$ is* time-consistent *if the corresponding dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t \in \mathcal{S}}$ is time-consistent and if the corresponding dynamic convex aggregation function $(\Lambda_{t,T})_{t \in \mathcal{S}}$ is time-consistent.*

In this section we study how the properties (r-TC), (a-TC1), (a-TC2) and the following properties of a dynamic convex systemic risk measure $(\rho_{t,T})_{t \in \mathcal{S}}$ depend on each other.

(s-TC1) Either all $\rho_{t,T}$ satisfy $\rho_{t,T}(\mathbb{R}^n I_{[t,\infty)}) = \mathbb{R}$ or all $\rho_{t,T}$ satisfy $\rho_{t,T}(\mathbb{R}^n I_{[t,\infty)}) = \mathbb{R}_+$.

(s-TC2) For every pair $s, t \in \mathcal{S}$ with $s \leq t$ and $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$,

$$\begin{aligned}&\bar{X} I_{[s,t)} = \bar{Y} I_{[s,t)} \text{ and } \rho_{t,T}(\bar{X}_u(\omega) I_{[t,\infty)}) \leq \rho_{t,T}(\bar{Y}_u(\omega) I_{[t,\infty)}) \\ &\text{for all } u \geq t \text{ and a.e. } \omega \in \Omega\end{aligned} \tag{12}$$

imply $\rho_{s,T}(\bar{X}_v(\omega) I_{[s,\infty)}) \leq \rho_{s,T}(\bar{Y}_v(\omega) I_{[s,\infty)})$ for all $v \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$.

(s-TC3) For every pair $s, t \in \mathcal{S}$ with $s \leq t$ and $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$,

$$\bar{X} I_{[s,t)} = \bar{Y} I_{[s,t)} \text{ and } \rho_{t,T}(\bar{X}) \leq \rho_{t,T}(\bar{Y})$$

imply $\rho_{s,T}(\bar{X}) \leq \rho_{s,T}(\bar{Y})$.

Suppose that $(\rho_{t,T})_{t \in \mathcal{S}}$ is a dynamic convex systemic risk measure $(\rho_{t,T})_{t \in \mathcal{S}}$ that satisfies (s-TC2). Since $\rho_{t,T}$ and $\rho_{s,T}$ also satisfy the preference consistency it follows for $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ that

$$\begin{aligned}&\bar{X} I_{[s,t)} = \bar{Y} I_{[s,t)} \text{ and } \rho_{t,T}(\bar{X}_u(\omega) I_{[t,\infty)}) \leq \rho_{t,T}(\bar{Y}_u(\omega) I_{[t,\infty)}) \quad \text{for all } u \geq t \text{ and a.e. } \omega \in \Omega \\ &\Rightarrow \rho_{s,T}(\bar{X}_v(\omega) I_{[s,\infty)}) \leq \rho_{s,T}(\bar{Y}_v(\omega) I_{[s,\infty)}) \quad \text{for all } v \in \mathbb{N}_0 \text{ and a.e. } \omega \in \Omega \\ &\Rightarrow \rho_{s,T}(\bar{X} I_{[s,\infty)}) \leq \rho_{s,T}(\bar{Y} I_{[s,\infty)}).\end{aligned}$$

The interpretation of property (s-TC3) is similar to the interpretation of (a-TC2). Let $\bar{X}$ and $\bar{Y}$ two loss processes that are equal up to time t and let the systemic risk at time t of one loss process be greater that the systemic risk of the other loss process. Then this inequality is still satisfied if we consider the systemic risk at time s.

Lemma 4.5. *Property (s-TC3) implies property (s-TC2).*

Proof. Consider $\bar{X},\bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ that satisfy (12). Then for a.e. $\omega \in \Omega$ and $v \geq t$ the processes $\bar{X}^{(\omega,v)},\bar{Y}^{(\omega,v)} \in \mathcal{R}^{\infty,n}_{s,T}$ defined by $\bar{X}^{(\omega,v)} := \bar{X}_v(\omega)I_{[t,\infty)}$ and $\bar{Y}^{(\omega,v)} := \bar{Y}_v(\omega)I_{[t,\infty)}$ satisfy $\bar{X}^{(\omega,v)}I_{[s,t)} = 0 \cdot 1_n I_{[s,t)} = \bar{Y}^{(\omega,v)}I_{[s,t)}$ and since (12) is satisfied, we obtain $\rho_{t,T}(\bar{X}^{(\omega,v)}) = \rho_{t,T}(\bar{X}_v(\omega)I_{[t,\infty)}) \leq \rho_{t,T}(\bar{Y}_v(\omega)I_{[t,\infty)}) = \rho_{t,T}(\bar{Y}^{(\omega,v)})$. (s-TC3) implies $\rho^0_{s,T}(\Lambda_{s,T}(\bar{X}_v(\omega))I_{[t,\infty)}) = \rho_{s,T}(\bar{X}^{(\omega,v)}) \leq \rho_{s,T}(\bar{Y}^{(\omega,v)}) = \rho^0_{s,T}(\Lambda_{s,T}(\bar{Y}_v(\omega))I_{[t,\infty)})$. Since $\rho^0_{s,T}$ satisfies the constancy property on $\mathbb{R}$ (respectively $\mathbb{R}_+$), it follows $\Lambda_{s,T}(\bar{X}_v(\omega)) \leq \Lambda_{s,T}(\bar{Y}_v(\omega))$, which implies

$$\rho_{s,T}(\bar{X}_v(\omega)I_{[s,\infty)}) = \rho^0_{s,T}(\Lambda_{s,T}(\bar{X}_v(\omega))I_{[s,\infty)}) \leq \rho^0_{s,T}(\Lambda_{s,T}(\bar{Y}_v(\omega))I_{[s,\infty)}) = \rho_{s,T}(\bar{Y}_v(\omega)I_{[s,\infty)}).$$

□

In the next lemma we prove that under certain conditions time-consistency of the dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$ and property (s-TC3) are equivalent.

Lemma 4.6. *Let $(\rho_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex systemic risk measure with corresponding dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$ and dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$. Moreover, assume that the following property is satisfied for all $s,t \in \mathcal{S}$ with $s \leq t$ and $\bar{X},\bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$:*

$$\rho^0_{s,T}(\Lambda_{t,T}(\bar{X})) \leq \rho^0_{s,T}(\Lambda_{t,T}(\bar{X})) \quad \Leftrightarrow \quad \rho^0_{s,T}(\Lambda_{s,T}(\bar{X})) \leq \rho^0_{s,T}(\Lambda_{s,T}(\bar{Y})) \tag{13}$$

Then $(\rho^0_{t,T})_{t\in\mathcal{S}}$ is time-consistent if and only if $(\rho_{t,T})_{t\in\mathcal{S}}$ satisfies (s-TC3).

Proof. "⇒" If $(\rho^0_{t,T})_{t\in\mathcal{S}}$ be time-consistent fix $s,t \in \mathcal{S}$ with $s \leq t$ and $\bar{X},\bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ with $\bar{X}I_{[s,t)} = \bar{Y}I_{[s,t)}$ and $\rho_{t,T}(\bar{X}) \leq \rho_{t,T}(\bar{Y})$. Note that the last property is equivalent to $\rho^0_{t,T}(\Lambda_{t,T}(\bar{X})) \leq \rho^0_{t,T}(\Lambda_{t,T}(\bar{Y}))$. Because the equality $\Lambda_{t,T}(\bar{X})I_{[s,t)} = \Lambda_{t,T}(\bar{Y})I_{[s,t)}$ is satisfied, it follows from (r-TC) that $\rho^0_{s,T}(\Lambda_{t,T}(\bar{X})) \leq \rho^0_{s,T}(\Lambda_{t,T}(\bar{Y}))$. Moreover, (13) implies $\rho_{s,T}(\bar{X}) = \rho^0_{s,T}(\Lambda_{s,T}(\bar{X})) \leq \rho^0_{s,T}(\Lambda_{s,T}(\bar{Y})) = \rho_{s,T}(\bar{Y})$.

"⇐" If (s-TC3) be satisfied, fix $s,t \in \mathcal{S}$ with $s \leq t$ and $X,Y \in \mathcal{R}^{\infty}_{s,T}$ with $XI_{[s,t)} = YI_{[s,t)}$ and $\rho^0_{t,T}(X) \leq \rho^0_{t,T}(Y)$. Then there exist $\bar{X},\bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ such that $\bar{X}I_{[s,t)} = \bar{Y}I_{[s,t)}$, $\Lambda_{t,T}(\bar{X}) = X$ and $\Lambda_{t,T}(\bar{Y}) = Y$. It follows from (s-TC3) that $\rho^0_{s,T}(\Lambda_{s,T}(\bar{X})) \leq \rho^0_{s,T}(\Lambda_{s,T}(\bar{Y}))$. Furthermore, (13) implies $\rho^0_{s,T}(X) = \rho^0_{s,T}(\Lambda_{t,T}(\bar{X})) \leq \rho^0_{s,T}(\Lambda_{t,T}(\bar{Y})) = \rho^0_{s,T}(Y)$. □

Example 4.7. (13), for instance, is satisfied for a dynamic positively homogeneous single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$ and a dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$ such that $\Lambda_{t,T} = q_t\Lambda$ for $q_t \in \mathbb{R}_+\backslash\{0\}$.

The following proposition characterizes dynamic convex aggregation functions in terms of the properties (s-TC1) and (s-TC2).

Proposition 4.8. *Let $(\rho_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex systemic risk measure with corresponding dynamic single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$ and dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$. Moreover, let $(\rho^0_{t,T})_{t\in\mathcal{S}}$ be time-consistent. Then $(\rho_{t,T})_{t\in\mathcal{S}}$ is time-consistent if and only if $(\rho_{t,T})_{t\in\mathcal{S}}$ satisfies (s-TC1) and (s-TC2).*

Proof. (s-TC1) is equivalent to (a-TC1) since $\rho_{t,T}(\bar{x}I_{[t,\infty)}) = \rho^0_{t,T}(\Lambda_{t,T}(\bar{x})I_{[t,\infty)}) = \Lambda_{t,T}(\bar{x})$ for all $\bar{x} \in \mathbb{R}^n$ and $t \in \mathcal{S}$.

Now, suppose that $(\rho_{t,T})_{t\in\mathcal{S}}$ satisfies (s-TC2). Let $\bar{x}, \bar{y} \in \mathbb{R}^n$ be such that $\Lambda_{t,T}(\bar{x}) \leq \Lambda_{t,T}(\bar{y})$. Moreover, fix $s, t \in \mathcal{S}$ with $s \leq t$ and define $\bar{Z} \in \mathcal{R}^{\infty,n}_{s,T}$ by $\bar{Z} := \bar{x}I_{[s,t)} + \bar{y}I_{[t,\infty)}$. Then $\Lambda_{t,T}(\bar{x}) \leq \Lambda_{t,T}(\bar{Z}_v(\omega))$ for all $v \geq t$ and a.e. $\omega \in \Omega$ and $\bar{Z}I_{[s,t)} = \bar{x}I_{[s,t)}$. Moreover, it follows from the monotonicity of $\rho^0_{t,T}$ that

$$\rho_{t,T}(\bar{x}I_{[t,\infty)}) = \rho^0_{t,T}(\Lambda_{t,T}(\bar{x})I_{[t,\infty)}) \leq \rho^0_{t,T}(\Lambda_{t,T}(\bar{Z}_v(\omega))I_{[t,\infty)}) = \rho_{t,T}(\bar{Z}_v(\omega)I_{[t,\infty)})$$

for all $v \geq t$ and a.e. $\omega \in \Omega$. Since (s-TC2) is satisfied, we obtain the inequality $\rho_{s,T}(\bar{x}I_{[s,\infty)}) \leq \rho_{s,T}(\bar{Z}_v(\omega)I_{[s,\infty)})$, which implies

$$\begin{aligned}&\rho^0_{s,T}(\Lambda_{s,T}(\bar{x})I_{[s,\infty)}) = \rho^0_{s,T}(\Lambda_{s,T}(\bar{x}I_{[s,\infty)})) \leq \rho^0_{s,T}(\Lambda_{s,T}(\bar{Z}_v(\omega)I_{[s,\infty)}))\\ &= \rho^0_{s,T}(\Lambda_{s,T}(\bar{y}I_{[s,\infty)})) = \rho^0_{s,T}(\Lambda_{s,T}(\bar{y})I_{[s,\infty)})\end{aligned}$$

for all $v \geq t$ and a.e. $\omega \in \Omega$. Finally, it follows $\Lambda_{s,T}(\bar{x}) \leq \Lambda_{s,T}(\bar{y})$ from the constancy property of $\rho^0_{s,T}$. This means that $(\rho^0_{t,T})_{t\in\mathcal{S}}$ is time-consistent.

To prove the other direction, let $(\rho_{t,T})_{t\in\mathcal{S}}$ be time-consistent and fix $\bar{X}, \bar{Y} \in \mathcal{R}^{\infty,n}_{s,T}$ with $\bar{X}I_{[s,t)} = \bar{Y}I_{[s,t)}$ and

$$\rho^0_{t,T}(\Lambda_{t,T}(\bar{X}_v(\omega))I_{[t,\infty)}) = \rho_{t,T}(\bar{X}_v(\omega)I_{[t,\infty)}) \leq \rho_{t,T}(\bar{Y}_v(\omega)I_{[t,\infty)}) = \rho^0_{t,T}(\Lambda_{t,T}(\bar{Y}_v(\omega))I_{[t,\infty)})$$

for all $v \geq t$ and a.e. $\omega \in \Omega$. The constancy property of $\rho^0_{t,T}$ on $\mathbb{R}$ (respectively $\mathbb{R}_+$) implies $\Lambda_{t,T}(\bar{X}I_{[t,\infty)}) \leq \Lambda_{t,T}(\bar{Y}I_{[t,\infty)})$. Since by assumption $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent, we have $\Lambda_{s,T}(\bar{X}I_{[t,\infty)}) \leq \Lambda_{s,T}(\bar{Y}I_{[t,\infty)})$. Moreover, the equality $\Lambda_{s,T}(\bar{X}I_{[s,t)}) = \Lambda_{s,T}(\bar{Y}I_{[s,t)})$ is trivial. It follows that $\Lambda_{s,T}(\bar{X}_v(\omega)) \leq \Lambda_{s,T}(\bar{Y}_v(\omega))$ for all $v \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$. From the monotonicity property of $\rho^0_{s,T}$ we obtain for all $v \in \mathbb{N}_0$ and a.e. $\omega \in \Omega$ that

$$\begin{aligned}&\rho_{s,T}(\bar{X}_v(\omega)I_{[s,\infty)}) = \rho^0_{s,T}(\Lambda_{s,T}(\bar{X}_v(\omega)I_{[s,\infty)})) = \rho^0_{s,T}(\Lambda_{s,T}(\bar{X}_v(\omega))I_{[s,\infty)})\\ &\leq \rho^0_{s,T}(\Lambda_{s,T}(\bar{Y}_v(\omega))I_{[s,\infty)}) = \rho^0_{s,T}(\Lambda_{s,T}(\bar{Y}_v(\omega)I_{[s,\infty)})) = \rho_{s,T}(\bar{Y}_v(\omega)I_{[s,\infty)})\end{aligned}$$

□

The following corollary sums up the previous results.

Corollary 4.9. *Let $(\rho_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex systemic risk measure with corresponding dynamic convex single-firm risk measure $(\rho^0_{t,T})_{t\in\mathcal{S}}$ and dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$. Moreover, assume that either all $\Lambda_{t,T}$ map into $\mathbb{R}$ or all $\Lambda_{t,T}$ map into $\mathbb{R}_+$. Then $(\rho_{t,T})_{t\in\mathcal{S}}$ is a time-consistent dynamic convex systemic risk measure if $(\rho^0_{t,T})_{t\in\mathcal{S}}$ is time-consistent and property* (13) *is satisfied.*

For the remaining part of this section we consider in detail (s-TC3), (s-TC2) and (a-TC2) and provide equivalent properties. In case of standard dynamic risk measures, Cheridito et al. (2006) (see Proposition 4.4) have proved the following result.

Proposition 4.10. *Let $(\rho^0_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex single-firm risk measure. Then the following statements are equivalent:*

1. $(\rho^0_{t,T})_{t\in\mathcal{S}}$ *is time-consistent.*
2. *For every pair* $s,t\in\mathcal{S}$ *with* $s\le t$ *and* $X\in\mathcal{R}^\infty_{s,T}$*, we have*

$$\rho^0_{s,T}(X)=\rho^0_{s,T}(XI_{[s,t)}+\rho^0_{t,T}(X)I_{[t,\infty)}).$$

For dynamic convex single firm risk measures we obtain the following result.

Proposition 4.11. *Let* $(\rho_{t,T})_{t\in\mathcal{S}}$ *be a dynamic convex systemic risk measure. Then the following statements are equivalent:*

1. $(\rho_{t,T})_{t\in\mathcal{S}}$ *satisfies (s-TC3).*
2. *For every pair* $s,t\in\mathcal{S}$ *with* $s\le t$ *and* $\bar X\in\mathcal{R}^{\infty,n}_{s,T}$*, we have*

$$\rho_{s,T}(\bar X)=\rho_{s,T}(\bar XI_{[s,t)}+f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar X))1_nI_{[t,\infty)})$$

Proof. 1. $\Rightarrow$ 2. : For $\bar X\in\mathcal{R}^{\infty,n}_{s,T}$ define the process $\bar Z:=\bar XI_{[s,t)}+f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar X))1_nI_{[t,\infty)}$. Because of property (s7), we obtain

$$\rho_{t,T}(\bar Z)=\rho_{t,T}(f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar X))1_nI_{[t,\infty)})=f_{\rho_{t,T}}(f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar X)))=\rho_{t,T}(\bar X).$$

Moreover, from (s-TC3) it follows directly that $\rho_{s,T}(\bar X)=\rho_{s,T}(\bar Z)$.

2. $\Rightarrow$ 1. : Fix $\bar X,\bar Y\in\mathcal{R}^{\infty,n}_{s,T}$ with $\bar XI_{[s,t)}=\bar YI_{[s,t)}$ and $\rho_{t,T}(\bar X)\le\rho_{t,T}(\bar Y)$. Since $f^{-1}_{\rho_{t,T}}$ and $\rho_{s,T}$ are monotone, we obtain

$$\begin{aligned}\rho_{s,T}(\bar X)&=\rho_{s,T}(\bar XI_{[s,t)}+f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar X))1_nI_{[t,\infty)})=\rho_{s,T}(\bar YI_{[s,t)}+f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar X))1_nI_{[t,\infty)})\\&\le\rho_{s,T}(\bar YI_{[s,t)}+f^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar Y))1_nI_{[t,\infty)})=\rho_{s,T}(\bar Y).\end{aligned}$$

□

The corresponding result to property (s-TC2) reads as follows. The proof is analogous to the proof of Proposition 4.11.

Proposition 4.12. *Let* $(\rho_{t,T})_{t\in\mathcal{S}}$ *be a dynamic convex systemic risk measure. Then the following statements are equivalent:*

1. $(\rho_{t,T})_{t\in\mathcal{S}}$ *satisfies (s-TC2).*
2. *For every pair* $s,t\in\mathcal{S}$ *with* $s\le t$ *and* $\bar X\in\mathcal{R}^{\infty,n}_{s,T}$*, we have for all* $v\in\mathbb{N}_0$ *and a.e.* $\omega\in\Omega$

$$\rho_{s,T}(\bar X_v(\omega)I_{[s,\infty)})=\rho_{s,T}(\bar Z_v(\omega)I_{[s,\infty)})$$

with $\bar Z_v(\omega):=\bar X_v(\omega)$ *for* $v<t$ *and* $\bar Z_v(\omega):=1_nf^{-1}_{\rho_{t,T}}(\rho_{t,T}(\bar X_v(\omega)I_{[t,\infty)}))$ *for* $v\ge t$.

Consider a dynamic convex aggregation function $(\Lambda_{t,T})_{t\in\mathcal{S}}$. If $\Lambda_{t,T}$, $t\in\mathcal{S}$, is $\mathbb{R}$-valued, then

$$\Lambda_{t,T}(f^{-1}_{\Lambda_{t,T}}(X)\,1_n)=f_{\Lambda_{t,T}}(f^{-1}_{\Lambda_{t,T}}(X))=X\quad\text{for all }X\in\mathcal{R}^\infty.$$

On the other hand, if $\Lambda_{t,T}$, $t\in\mathcal{S}$, is $\mathbb{R}_+$-valued, then the inverse function is given by $f^{-1}_{\Lambda_{t,T}}:\mathbb{R}_+\to[b_{t,T},\infty)$ for $b_{t,T}\ge0$. Moreover, we have

$$\Lambda_{t,T}(f^{-1}_{\Lambda_{t,T}}(X^+)1_n)=f_{\Lambda_{t,T}}(f^{-1}_{\Lambda_{t,T}}(X^+))=X^+\quad\text{for all }X\in\mathcal{R}^\infty.$$

Proposition 4.13. *Let $(\Lambda_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex aggregation function. Then the following statements are equivalent:*

1. *$(\Lambda_{t,T})_{t\in\mathcal{S}}$ satisfies (a-TC2).*
2. *For every pair $s,t \in \mathcal{S}$ with $s \le t$ and $\bar{X} \in \mathcal{R}^{\infty,n}_{s,T}$, we have*

$$\Lambda_{s,T}(\bar{X}I_{[t,\infty)}) = \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{X}))1_n I_{[t,\infty)}).$$

3. *For every pair $s,t \in \mathcal{S}$ with $s \le t$ and $\bar{x} \in \mathbb{R}^n$, we have*

$$\Lambda_{s,T}(\bar{x}) = \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))1_n).$$

Proof. 1. $\Rightarrow$ 2. : Fix $\bar{X} \in \mathcal{R}^{\infty,n}_{s,T}$ and set $\bar{Z} := f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{X}))1_n$. Because of the preliminary remarks, we obtain $\Lambda_{t,T}(\bar{Z}I_{[t,\infty)}) = \Lambda_{t,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{X}I_{[t,\infty)}))1_n) = \Lambda_{t,T}(\bar{X}I_{[t,\infty)})$ and property (a-TC2) implies $\Lambda_{s,T}(\bar{X}I_{[t,\infty)}) = \Lambda_{s,T}(\bar{Z}I_{[t,\infty)})$.

2. $\Rightarrow$ 3. : Let $\bar{x} \in \mathbb{R}^n$ and consider the process $\bar{x}I_{[s,\infty)} \in \mathcal{R}^{\infty,n}_{s,T}$. Since the equality in 2. is satisfied and since $\Lambda_{s,T}$, $\Lambda_{t,T}$ and $f^{-1}_{\Lambda_{t,T}}$ are measurable, it follows for all $v \ge t$ and a.e. $\omega \in \Omega$ that

$$\begin{aligned}
&\Lambda_{s,T}(\bar{x}) = \Lambda_{s,T}(\bar{x}I_{[s,\infty)}(v,\omega)) = \Lambda_{s,T}(\bar{x}I_{[s,\infty)})(v,\omega) = \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}I_{[s,\infty)}))1_n I_{[t,\infty)})(v,\omega)\\
&= \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}I_{[s,\infty)}(v,\omega)))1_n I_{[t,\infty)})(v,\omega) = \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))1_n).
\end{aligned}$$

3. $\Rightarrow$ 1. : We will prove the equivalent property (a-TC2'). Fix $\bar{x},\bar{y} \in \mathbb{R}^n$ with $\Lambda_{t,T}(\bar{x}) \le \Lambda_{t,T}(\bar{y})$. The equality in 3. and the monotonicity properties of $\Lambda_{s,T}$ and $f^{-1}_{\Lambda_{t,T}}$ imply $\Lambda_{s,T}(\bar{x}) = \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))1_n) \le \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{y}))1_n) = \Lambda_{s,T}(\bar{y})$. □

4.2 Examples of time-consistent dynamic aggregation functions

By definition, a time-consistent dynamic convex systemic risk measure is a composition of a time-consistent dynamic single-firm risk measure and a time-consistent dynamic aggregation function. Examples for time-consistent dynamic single-firm risk measures can be found in Cheridito et al. (2006) and Cheridito and Kupper (2011). In this section we provide several examples for time-consistent dynamic convex aggregation functions.

Proposition 4.14. *Let $(\Lambda_{t,T})_{t\in\mathcal{S}}$ be a dynamic convex aggregation function such that either all $\Lambda_{t,T}$ map into $\mathbb{R}$ or all $\Lambda_{t,T}$ map into $\mathbb{R}_+$.*

1. *If all $\Lambda_{t,T}$, $t \in \mathcal{S}$, are $\mathbb{R}$-valued, then $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent if and only if $f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))$ does not depend on $s \in \mathcal{S}$ for all $\bar{x} \in \mathbb{R}^n$.*
2. *If all $\Lambda_{t,T}$, $t \in \mathcal{S}$, are $\mathbb{R}_+$-valued, then $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent if and only if the following properties are satisfied:*
   - *(a) We have $b_{t,T} \le b_{s,T}$ for all $s,t \in \mathcal{S}$ with $s \le t$.*
   - *(b) For all $s,t \in \mathcal{S}$ with $s \le t$, the following conditions hold:*
     - *i. $\Lambda_{t,T}(\bar{x}) > 0$ and $\Lambda_{s,T}(\bar{x}) = 0$ for $\bar{x} \in \mathbb{R}^n$ imply $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) \le f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))$.*

ii. $\Lambda_{t,T}(\bar{x}) > 0$ *and* $\Lambda_{s,T}(\bar{x}) > 0$ *for* $\bar{x} \in \mathbb{R}^n$ *imply* $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) = f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))$.

In particular, $(\Lambda_{t,T})_{t\in\mathcal{S}}$ *is time-consistent if* $\Lambda_{t,T} = r_t\Lambda$ *for a convex aggregation function* Λ *and* $r_t \in \mathbb{R}_+\backslash\{0\}$.

Proof. Part 1: First, consider a time-consistent $(\Lambda_{t,T})_{t\in\mathcal{S}}$, suppose that all $\Lambda_{t,T}$, $t \in \mathcal{S}$, are $\mathbb{R}$-valued and let $\bar{x} \in \mathbb{R}^n$. Proposition 4.13 yields that

$$\Lambda_{s,T}(\bar{x}) = \Lambda_{s,T}(f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))1_n) = \Lambda_{s,T}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))1_n) \quad \text{for all } s,t \in \mathcal{S} \text{ with } s \leq t.$$

Furthermore, because of the $f_{\Lambda_{s,T}}$-constancy property we have

$$\Lambda_{s,T}(\bar{x}) = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))) = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))). \tag{14}$$

Since $f_{\Lambda_{s,T}} : \mathbb{R} \to \mathbb{R}$ is bijective, we obtain $f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x})) = f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))$. Hence, the "only if" part is satisfied. The "if" part is trivial.

Part 2: Suppose that all $\Lambda_{t,T}$ are $\mathbb{R}_+$-valued. Moreover, let $(\Lambda_{t,T})_{t\in\mathcal{S}}$ be time-consistent and consider $\bar{x} \in \mathbb{R}^n$ and $s,t \in \mathcal{S}$ with $s \leq t$. Note that either $\Lambda_{t,T}(\bar{x}) = 0$ or $\Lambda_{t,T}(\bar{x}) > 0$. In case of $\Lambda_{t,T}(\bar{x}) = 0$, it follows from Equation (14) that $\Lambda_{s,T}(\bar{x}) = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))) = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{t,T}}(0))$. Since $f^{-1}_{\Lambda_{t,T}}(0) = b_{t,T}$, we obtain $\Lambda_{s,T}(\bar{x}) = f_{\Lambda_{s,T}}(b_{t,T}) = 0$, which implies that $b_{t,T} \leq b_{s,T}$. Now, let us consider the case $\Lambda_{t,T}(\bar{x}) > 0$. Because $f_{\Lambda_{t,T}}|_{[b_{t,T},\infty)}$ is strictly increasing, we have $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) > b_{t,T}$. If $\Lambda_{t,T}(\bar{x}) > 0$ and $\Lambda_{s,T}(\bar{x}) = 0$, then $0 = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))) = f_{\Lambda_{s,T}}(f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})))$. It follows that $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) \leq b_{s,T} = f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))$. If $\Lambda_{t,T}(\bar{x}) > 0$ and $\Lambda_{s,T}(\bar{x}) > 0$, then we obtain from the strict monotonicity of $f^{-1}_{\Lambda_{s,T}}$ that $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) > b_{s,T}$ and $f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x})) > b_{s,T}$. Furthermore, bijectivity and (14) imply $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) = f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))$. Thus, the "only if" part of the assertion is satisfied. The "if" part is again trivial.

To the last assertion: Suppose that $\Lambda_{t,T} = r_t\Lambda$ for $r_t \in \mathbb{R}_+\backslash\{0\}$. If all $\Lambda_{t,T}$ map into $\mathbb{R}$, then

$$f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) = f^{-1}_{\Lambda}(\Lambda_{t,T}(\bar{x})/r_t) = f^{-1}_{\Lambda}(r_t\Lambda(\bar{x})/r_t) = f^{-1}_{\Lambda}(\Lambda(\bar{x})) \quad \text{for } \bar{x} \in \mathbb{R}^n.$$

Moreover, $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent because of the first part of the proposition.

If all $\Lambda_{t,T}$ map into $\mathbb{R}_+$, then $f_{\Lambda_{t,T}}(a) = r_t f_{\Lambda}(a)$ for all $a \in \mathbb{R}$ and $f^{-1}_{\Lambda_{t,T}}(a) = f^{-1}_{\Lambda}(a/r_t)$ for all $t \in \mathcal{S}$ and $a \in \mathbb{R}_+$. Furthermore, we have $b = b_{t,T}$ for all $t \in \mathcal{S}$ and $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) = f^{-1}_{\Lambda}(\Lambda(\bar{x}))$ for $\bar{x} \in \mathbb{R}^n$. It follows for $\Lambda_{t,T}(\bar{x}) > 0$ and $\Lambda_{s,T}(\bar{x}) = 0$ that $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x})) = f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x})) = b_{s,T}$. Finally, $(\Lambda_{t,T})_{t\in\mathcal{S}}$ is time-consistent because of the second part of the proposition. □

In the following lemma we analyze specific dynamic convex aggregation functions $(\Lambda_{t,T})_{t\in\mathcal{S}}$.

Lemma 4.15. *Let* $(\Lambda_{t,T})_{t\in\mathcal{S}}$ *be a dynamic convex aggregation function with*

$$\Lambda_{t,T}(\bar{x}) := \sum_{i=1}^{n} g_{t,T}((\bar{x}_i - b_{t,T})^+) \quad \textit{for } \bar{x} \in \mathbb{R}^n \tag{15}$$

where $b_{t,T} \in \mathbb{R}_+$ *and* $g_{t,T} : \mathbb{R}_+ \to \mathbb{R}_+$ *are convex and strictly increasing functions with* $g_{t,T}(0) = 0$. *Then* $(\Lambda_{t,T})_{t\in\mathcal{S}}$ *is time-consistent if and only if* $b_{t,T} = b$ *for* $t \in \mathcal{S}$ *and* $f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x})) = f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))$ *for all* $\bar{x} \in \mathbb{R}$ *and* $s,t \in \mathcal{S}$.

Proof. Consider a time-consistent dynamic convex aggregation $(\Lambda_{t,T})_{t\in\mathcal{S}}$ and let $\bar{x}\in\mathbb{R}^n$ and $s,t\in\mathcal{S}$ with $s<t$. We know from the second part in Proposition 4.14 that $b_{t,T}\le b_{s,T}$. Now, assume that $b_{t,T}<b_{s,T}$. Then $f_{\Lambda_{r,T}}(x)=ng_{r,T}((x-b_{r,T})^+)$ for $x\in\mathbb{R}$ and $f^{-1}_{\Lambda_{r,T}}(x)=g^{-1}_{r,T}(x/n)+b_{r,T}$ for $x\in\mathbb{R}_+$ and $r\in\mathcal{S}$, and hence

$$f^{-1}_{\Lambda_{r,T}}(\Lambda_{r,T}(\bar{x}))=g^{-1}_{r,T}(\Lambda_{r,T}(\bar{x})/n)+b_{r,T}=g^{-1}_{r,T}\left(\sum_{i=1}^{n}g_{r,T}((\bar{x}^i-b_{r,T})^+)/n\right)+b_{r,T}\quad\text{for all } r\in\mathcal{S}. \tag{16}$$

Consider $\bar{y},\bar{z}\in\mathbb{R}^n$ such that $\bar{y}_1=b_{s,T}+1$, $\bar{y}_2=\ldots=\bar{y}_n=b_{t,T}$ and $\bar{z}_1=b_{s,T}+1$, $\bar{z}_2=b_{t,T}+c$, $\bar{z}_2=\ldots=\bar{z}_n=b_{t,T}$ for $c:=(b_{s,T}-b_{t,T})/2>0$. Then $\Lambda_{s,T}(\bar{y}),\Lambda_{s,T}(\bar{z})>0$ and $\Lambda_{t,T}(\bar{y}),\Lambda_{t,T}(\bar{z})>0$ and because of the second part in Proposition 4.14, we obtain

$$\begin{aligned}
&g^{-1}_{s,T}(g_{s,T}(1)/n)+b_{s,T}=g^{-1}_{t,T}(g_{t,T}(b_{s,T}+1-b_{t,T})/n)+b_{t,T}\quad\text{and}\\
&g^{-1}_{s,T}(g_{s,T}(1)/n)+b_{s,T}=g^{-1}_{t,T}([g_{t,T}(b_{s,T}+1-b_{t,T})+g_{t,T}(d)]/n)+b_{t,T}.
\end{aligned}$$

But since $g_{t,T}(c)>0$ and $g^{-1}_{t,T}:\mathbb{R}^+\to\mathbb{R}^+$ is strictly increasing, this is a contradiction.

Finally, we will prove that $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))=f^{-1}_{\Lambda_{s,T}}(\Lambda_{s,T}(\bar{x}))=b$ in case of $\Lambda_{t,T}(\bar{x})>0$ and $\Lambda_{s,T}(\bar{x})=0$ for $\bar{x}\in\mathbb{R}^n$. We know from the second part of Proposition 4.14 that $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))\le b$. Moreover, if $f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))<b$, then (16) implies

$$f^{-1}_{\Lambda_{t,T}}(\Lambda_{t,T}(\bar{x}))=g^{-1}_{t,T}\left(\sum_{i=1}^{n}g_{t,T}((\bar{x}_i-b)^+)/n\right)+b<b,$$

which is a contradiction.

The other direction of the proof is trivial. □

Let us consider the convex aggregation functions introduced in Kromer et al. (2014). In combination with the previous results we obtain the following examples for dynamic time-consistent aggregation functions.

Example 4.16. Define for $t\in\mathcal{S}$ the following convex aggregation functions

$$\Lambda^{\text{sum}}_{t,T}(\bar{x}):=r_t\sum_{i=1}^{n}\bar{x}_i\qquad\qquad \Lambda^{\text{sumexp}}_{t,T}(\bar{x}):=r_t\sum_{i=1}^{n}(\exp(\gamma\bar{x}_i^+)-1),\quad \gamma>0$$

$$\Lambda^{\text{loss}}_{t,T}(\bar{x}):=r_t\sum_{i=1}^{n}\bar{x}_i^+,\qquad\qquad \Lambda^{\text{plin}}_{t,T}(\bar{x}):=r_t\sum_{i=1}^{n}\lambda(\bar{x}_i)$$

$$\Lambda^{b,\text{loss}}_{t,T}(\bar{x}):=r_t\sum_{i=1}^{n}(\bar{x}_i-b)^+,\quad b\in\mathbb{R}_+$$

where $\bar{x}\in\mathbb{R}^n$, $r_t>0$ for each $t\in\mathcal{S}$ and

$$\lambda(x)=\begin{cases}0 & \text{for } x<0\\ ax & \text{for } 0\le x<c\\ b(x-c)+ac & \text{for } x\ge c\end{cases}$$

with $0 < a < b$ and $c > 0$. Then $(\Lambda^{\text{sum}}_{t,T})_{t\in\mathcal{S}}$, $(\Lambda^{b,\text{loss}}_{t,T})_{t\in\mathcal{S}}$, $(\Lambda^{\text{sumexp}}_{t,T})_{t\in\mathcal{S}}$ and $(\Lambda^{\text{plin}}_{t,T})_{t\in\mathcal{S}}$ are time-consistent dynamic convex aggregation functions.

Example 4.17. Let the convex aggregation function $\Lambda^{\exp}_{t,T}$, $t \in \mathcal{S}$, be given by

$$\Lambda^{\exp}_{t,T}(\bar{x}) := \exp\left(r_t \sum_{i=1}^{n} \bar{x}_i^+\right) - 1 \quad \text{for } \bar{x} \in \mathbb{R}^n \text{ and } r_t > 0.$$

Then $f^{-1}_{\Lambda^{\exp}_{t,T}}(\Lambda^{\exp}_{t,T}(\bar{x})) = \sum_{i=1}^{n} \bar{x}_i^+/n$ for $\bar{x} \in \mathbb{R}^n$ and the second part in Proposition 4.14 yields that $(\Lambda^{\exp}_{t,T})_{t\in\mathcal{S}}$ is a time-consistent dynamic convex aggregation function.

Example 4.18. For $k > 1$ let the convex aggregation function $\Lambda^{[k]}_{t,T}$, $t \in \mathcal{S}$, be given by

$$\Lambda^{[k]}_{t,T}(\bar{x}) := \left(r_t \sum_{i=1}^{n} \bar{x}_i^+\right)^k \quad \text{for } \bar{x} \in \mathbb{R}^n \text{ and } r_t > 0.$$

Again, we have $f^{-1}_{\Lambda^{[k]}_{t,T}}(\Lambda^{[k]}_{t,T}(\bar{x})) = \sum_{i=1}^{n} \bar{x}_i^+/n$, and hence $(\Lambda^{[k]}_{t,T})_{t\in\mathcal{S}}$ is time-consistent.

In our approach we have considered time-consistent dynamic single-firm risk measures $\rho^0_{t,T} : \mathcal{R}^\infty_{t,T} \to L^\infty_t$ and time-consistent convex aggregation functions $\Lambda_{t,T}$ as functions from $\mathbb{R}^n$ to $\mathbb{R}$. In this setting, we have studied how this concepts relate to the time-consistency of the corresponding dynamic systemic risk measure and obtained a characterization, see Proposition 4.8. Note that in our setting all dynamic convex aggregation functions are functions from $\mathbb{R}^n$ to $\mathbb{R}$ which, in a certain sense, limits the possibilities to construct dynamic convex aggregation functions that satisfy the needed requirements for the time-consistency of a dynamic systemic risk measure.

Nevertheless, there still exist several interesting examples of time-consistent dynamic convex aggregation functions, see Lemma 4.15 and Examples 4.16-4.18, in this rather simple setting of functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Furthermore, one could bring forward the argument that it also stands to reason to introduce time-consistent dynamic systemic risk measures $(\rho_{t,T})_{t\in\mathcal{S}}$ in terms of a time-consistent dynamic single-firm risk measure and a specific (static) aggregation function, which is fixed at the beginning of the time period and then used throughout, i.e. $\rho_{t,T} = \rho^0_{t,T} \circ \Lambda$. In this case, the dynamics of the systemic risk measure would be captured by the dynamics of the underlying single-firm risk measure.

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