Representation of BSDE-based Dynamic Risk Measures and Dynamic Capital Allocations

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Abstract

In this short paper we provide a new representation result for dynamic capital allocations and dynamic convex risk measures that are based on backward stochastic differential equations. We derive this representation from a classical differentiability result for backward stochastic differential equations and the full allocation property of the Aumann-Shapley allocation. The representation covers BSDE-based dynamic convex and dynamic coherent risk measures. The results are applied to derive a representation for the dynamic entropic risk measure. Our results are also applicable in a specific way to the static case, where we are able to derive a new representation result for static convex risk measures that are Gâteaux-differentiable.

Keywords Dynamic risk measure, dynamic risk capital allocation, backward stochastic differential equation, gradient allocation, Aumann-Shaley allocation, dynamic entropic risk measure.

1 Introduction

The study of risk measures began in a static environment with the papers of Artzner et al. (1999) and their introduction of coherent risk measures. These were generalized later on by convex risk measures that were introduced in Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2002). An important application of risk measures are capital applications. These are needed if one is interested in the decomposition of a portfolio wide risk capital into a sum of risk contributions by the respective subportfolios. The gradient allocation, which is the Gâteaux-derivative of the underlying risk measure at portfolio X in the direction of the subportfolio Y, has proved to be the most studied and applied capital allocation method for coherent risk measures, see for instance Cherny and Orlov (2011), Denault (2001), Kalkbrener (2005), Kromer and Overbeck (2013b) and Tasche (2004). This is mainly justified by several desirable structural properties of the gradient allocation. For convex risk measures, a useful method that is strongly connected to the gradient allocation is the Aumann-Shapley allocation, see Aumann and Shapley (1974) and Billera et al. (1978).

To incorporate information structure over time static risk measures were extended to a dynamic setting in Barrieu and El Karoui (2009), Cheridito et al. (2006), Detlefsen and Scandolo (2005), Jobert and Rogers (2008) and many others. Dynamic risk measures have been studied in several different frameworks. We are interested in a specific class of dynamic risk measures, namely dynamic risk measures that arise as solutions to certain types of backward stochastic differential equations (or BSDEs for short). BSDEs in general have been studied, among others, in Briand et al. (2003), Kobylanski (2000), Ma and Yong (1999) and Pardoux and Peng (1990). There are numerous applications of BSDEs in finance that can be found in Barrieu and El Karoui (2009), El Karoui et al. (1997), Mania and Schweizer (2005), Peng (2004) and Rosazza Gianin (2006). For the connection of BSDEs to dynamic risk measures we refer in particular to Barrieu and El Karoui (2009). Dynamic capital allocations have been studied in Cherny (2009) in discrete time and in Kromer and Overbeck (2013a) in continuous time. A classical differentiability result for BSDEs from Ankirchner et al. (2007) will enable us to derive a representation result for the BSDE-based dynamic gradient allocation. This result, together with the full allocation property of the Aumann-Shapley allocation leads to a representation of BSDE-based dynamic risk measures. For the dynamic coherent case the risk measures can be represented in terms of conditional expectations with respect to an equivalent probability measure. This probability measure is constructed by a Girsanov-type change of measure argument, where the corresponding density process is based on the derivative of the generator of the underlying BSDE. Our results are also applicable in a specific way to the static case, where we are able to derive a new representation result for static convex risk measures that are Gâteaux-differentiable.

The outline of the paper is as follows. The necessary notation and the required definitions of dynamic risk measures, BSDEs and capital allocations are presented in Section 2. In Section 3 we provide a representation result for the BSDE-based dynamic gradient allocation. This result leads to the representation result for BSDE-based dynamic convex and coherent risk measures in Section 4. The difference between the two representations is that in the convex case the representation involves a process that depends on all portfolio weights γ, $\gamma \in [0,1]$, whereas in the coherent case this process becomes a density process that depends only on the portfolio weight $\gamma = 1$. As a conclusion of the paper we study in Section 5 the dynamic entropic risk measure as a representative example for the class of dynamic convex risk measures. We apply our results to this dynamic risk measure and provide the corresponding representation. Furthermore we will see that the full allocation property can be used in the same way to obtain representations for (static) Gâteaux-differentiable convex and coherent risk measures. In this regard we apply our results to the (static) entropic risk measure and to (static) transformed norm risk measures that have been introduced in Cheridito and Li (2008).

2 Preliminaries

In this section we will collect definitions and notations that will be needed in the following sections. We will start with a short introduction of (static) coherent and convex risk measures and the gradient allocation and then move on to dynamic risk measures and BSDEs. Consider $\mathcal{X}$ as some space of financial positions X that are defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Usually $\mathcal{X}$ is considered as the space $L^\infty(\Omega, \mathcal{G}, \mathbb{P})$ of essentially bounded random variables with norm $\|X\|_\infty = \operatorname{ess\,sup}|X|$ or the space $L^p(\Omega, \mathcal{G}, \mathbb{P})$, $1 \leq p < \infty$ of p-integrable random variables. A mapping $\rho : \mathcal{X} \to (-\infty, \infty]$ is called (static) monetary risk measure if it satisfies the properties (S1)-(S2) below. If it satisfies (S1)-(S3), it is called a (static) convex risk measure. If it fulfills (S1)-(S4), it is a (static) coherent risk measure, see Artzner et al. (1999) and Föllmer and Schied (2002).

(S1) $\rho(X) \geq \rho(Y)$ for $X \leq Y$,

(S2) $\rho(X + m) = \rho(X) - m$ for all $X \in \mathcal{X}$ and $m \in \mathbb{R}$,

(S3) $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for all $X, Y \in \mathcal{X}$ and $\lambda \in (0, 1)$

(S4) $\rho(\lambda X) = \lambda\rho(X)$ for all $X \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+$.

We will denote the domain of a convex risk measure ρ with $\operatorname{dom}\rho = \{X \in \mathcal{X} \mid \rho(X) < \infty\}$. ρ is a *proper* convex risk measure if the domain of ρ is nonempty. If ρ is continuous at $X \in \operatorname{dom}\rho$ it is known that the subdifferential of ρ at X, $\partial\rho(X)$, is a singleton if and only if ρ is Gâteaux-differentiable, see for instance Zalinescu (2002). This means that $\partial\rho(X) = \{\nabla\rho(X)\}$ and for any $Y \in \mathcal{X}$

$$\nabla_Y \rho(X) := \mathbb{E}(\nabla\rho(X)Y) = \lim_{h\to 0} \frac{\rho(X + hY) - \rho(X)}{h} = \frac{d}{du}\rho(X + uY)\bigg|_{u=0}. \tag{2.1}$$

is the Gâteaux-derivative of ρ at X in the direction of Y.

If we suppose that $X \in \mathcal{X}$ represents a portfolio with risk $\rho(X)$ and the portfolio X consists of the subportfolios X_i, $i = 1, \ldots, n$,

$$X = \sum_{i=1}^{n} X_i,$$

then we are interested in the allocation of the risk capital of the whole portfolio to the constituent parts of the portfolio. The objective is thus to determine the risk contributions of each X_i, $i = 1, \ldots, n$, to the risk capital of X. For a Gâteaux-differentiable risk measure ρ the (static) gradient allocation principle is defined as the Gâteaux-derivative at X in the direction of X_i, $i = 1, \ldots, n$. The (static) Aumann-Shapley allocation, on the other hand, is defined as

$$\overline{\nabla_{X_i}\rho(X)} := \int_0^1 \nabla_{X_i}\rho(\gamma X)d\gamma, \quad i = 1, \ldots, n. \tag{2.2}$$

If ρ satisfies property (S4), which means that it is positively homogeneous, then the Aumann-Shapley allocation reduces to the gradient allocation. If ρ is positively homogeneous, the gradient allocation satisfies the full allocation property. This means that the sum of the risk contributions adds up to the risk of the portfolio,

$$\rho(X) = \sum_{i=1}^{n} \nabla_{X_i}\rho(X), \tag{2.3}$$

see for instance Denault (2001), Kalkbrener (2005) and Tasche (2004). The Aumann-Shapley allocation itself does not require ρ to be positively homogeneous to satisfy the full allocation property. Thus it can be used in conjunction with Gâteaux-differentiable convex risk measures that do not satisfy property (S4). We will go more into detail on the full allocation property of the Aumann-Shapley allocation in Section 4.

Let us now move on to dynamic risk measures and their connection to BSDEs. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $W = (W(t))_{t\geq 0}$ be a d-dimensional Brownian motion. Denote by $\{\mathcal{F}_t\}_{t\geq 0}$ the natural Brownian filtration augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. Denote by $\mathbb{E}$ the expected value with respect to $\mathbb{P}$. For the definition of BSDE-based dynamic risk measures we will follow Barrieu and El Karoui (2009) and we will need the following spaces. Denote by $\mathcal{R}^\infty(0, T)$ the space of bounded measurable processes and by $\mathbb{L}^2([0, T])$ the space of all progressively measurable processes $X(\cdot)$ such that

$$\mathbb{E}\left[\int_0^T |X(s)|^2 ds\right] < \infty$$

We will denote by $L^\infty(\mathcal{F}_T)$ the space of $\mathcal{F}_T$ measurable, essentially bounded random variables. For any process $X(\cdot)$ with existing quadratic variation let $\mathcal{E}(X)(\cdot)$ be the strictly positive adapted process

$$\mathcal{E}(X)(t) = \exp\left\{X(t) - \frac{1}{2}\langle X(t)\rangle\right\}, \quad t \in [0, T].$$

Denote by $C_b^\infty(\mathbb{R}^n)$ the set of C^∞ functions $f : \mathbb{R}^n \to \mathbb{R}$ which are bounded and have bounded derivatives of all orders. Let $\mathcal{S}$ be the class of smooth functionals, i.e. random variables of the form $F(\omega) = f(W(t_1, \omega), \ldots, W(t_n, \omega))$, where $(t_1, \ldots, t_n) \in [0, T]^n$. The Malliavin derivative $DF(\omega)$ of the smooth random variable F is defined as the $(\mathbb{L}^2([0, T]))^d$-valued random variable

$$D_t^i F(\omega) = \sum_{j=1}^{n} \frac{\partial}{\partial x^j} f(W(t_1, \omega), \ldots, W(t_n, \omega)) 1_{[0,t_j]}(t), \quad t \in [0, T], \, i = 1, \ldots, d.$$

The space $\mathbb{D}_{1,1}$ denotes the Banach space which is the closure of $\mathcal{S}$ under $\|\cdot\|_{1,1}$, where

$$\|\xi\|_{1,1} := \mathbb{E}\left[|\xi| + \left(\sum_{i=1}^{d} \|D^i\xi\|^2\right)^{1/2}\right],$$

and $\|\cdot\|$ denotes the $\mathbb{L}^2([0,T])$-norm, see Nualart (2006) and Ocone and Karatzas (1991). Denote by $BMO(\mathbb{P})$ the class of processes $Z(\cdot)$ such that there exists a constant $C \in \mathbb{R}_+$ such that, for each $t \leq T$

$$\mathbb{E}\left[\int_t^T |Z(s)|^2 ds \middle| \mathcal{F}_t\right] \leq C \quad \mathbb{P}\text{-a.s.}$$

Now, let us consider the backward stochastic differential equation (BSDE)

$$Y^\xi(t) = -\xi - \int_t^T Z^\xi(s)dW(s) + \int_t^T g(s, Z^\xi(s))ds \tag{2.4}$$

with generator g which is of the form

$$g(t,z) = l(t,z) + \alpha|z|^2 \tag{2.5}$$

where $(l(t,0); 0 \leq t \leq T)$ is progressively measurable, satisfies

$$\mathbb{E}\left[\int_0^T |l(s,0)|^2 ds\right] < \infty$$

and l is Lipschitz continuous in z. We know from Kobylanski (2000) that if $\xi \in L^\infty(\mathcal{F}_T)$ and the generator g is of the form (2.5) then there exists a solution $(Y(\cdot), Z(\cdot)) \in \mathcal{R}^\infty \times \mathbb{L}^2$ to (2.4). Thus, in this context, we will consider a BSDE-based dynamic risk measure as a map from $L^\infty(\mathcal{F}_T)$ to $\mathcal{R}^\infty(0,T)$ that is based on the following properties.

(D1) Convexity: For any stopping times $S \leq T$, for any $(\xi, \psi) \in L^\infty(\mathcal{F}_T) \times L^\infty(\mathcal{F}_T)$ and for any $0 \leq \lambda \leq 1$

$$\rho(S; \lambda\xi + (1-\lambda)\psi) \leq \lambda\rho(S;\xi) + (1-\lambda)\rho(S;\psi) \quad \mathbb{P}\text{-a.s.}$$

(D2) Monotonicity: For any stopping times S, T and any $(\xi, \psi) \in L^\infty(\mathcal{F}_T) \times L^\infty(\mathcal{F}_T)$ such that $\xi \geq \psi$ $\mathbb{P}$-a.s.

$$\rho(S;\xi) \leq \rho(S;\psi) \quad \mathbb{P}\text{-a.s.}$$

(D3) Translation property: For any stopping times S, T, any $\psi \in L^\infty(\mathcal{F}_S)$ and any $\xi \in L^\infty(\mathcal{F}_T)$

$$\rho(S; \xi + \psi) = \rho(S;\xi) - \psi \quad \mathbb{P}\text{-a.s.}$$

(D4) Time consistency: For any three bounded stopping times $S \leq T \leq U$ and for any $\xi \in L^\infty(\mathcal{F}_U)$

$$\rho(S;\xi) = \rho(S; -\rho(T;\xi)) \quad \mathbb{P}\text{-a.s.}$$

(D5) Arbitrage freedom: For any stopping times $S \leq T$ and any $(\xi, \psi) \in L^\infty(\mathcal{F}_T) \times L^\infty(\mathcal{F}_T)$ such that $\xi \geq \psi$

$$\rho(S;\xi) = \rho(S;\psi) \quad \text{on} \quad A_S = \{S < T\} \quad \Rightarrow \quad \xi = \psi \quad \mathbb{P}\text{-a.s. on } A_S.$$

(D6) Positive homogeneity: For any stopping times $S \leq T$, for any $\lambda_S \geq 0$, $\lambda_S \in L^\infty(\mathcal{F}_S)$ and for any $\xi \in L^\infty(\mathcal{F}_T)$

$$\rho(S;\lambda_S\xi) = \lambda_S \rho(S;\xi) \quad \mathbb{P}\text{-a.s.}$$

With the above properties we are now able to define dynamic convex and dynamic coherent risk measures.

Definition 2.1 *A BSDE-based dynamic convex risk measure $\rho(\cdot)$ is a map from $L^\infty(\mathcal{F}_T)$ to $\mathcal{R}^\infty(0,T)$ that satisfies the properties (D1)-(D5). If $\rho(\cdot)$ additionally satisfies property (D6) then it is called a BSDE-based dynamic coherent risk measure.*

If l is convex in z then we know from Theorem 6.7 in Barrieu and El Karoui (2009) that the first component of the maximal solution to BSDE (2.4),

$$\rho(t;\xi) = Y^\xi(t) \quad \text{for all } t \in [0,T],$$

is a BSDE-based dynamic convex risk measure. Furthermore $\rho(\cdot)$ is a BSDE-based dynamic coherent risk measure if $g(t,z) = l(t,z)$ and l is convex and positively homogeneous in z.

As can be seen from the definition of the gradient allocation in (2.1) and the Aumann-Shapley allocation in (2.2) we will need a classical differentiability result for BSDEs to be able to study BSDE-based dynamic versions of these allocation methods. For this purpose we will use the BSDE-differentiability result from Ankirchner et al. (2007). In this paper the authors consider parameter dependent BSDEs of the following type

$$Y^u(t) = \xi(u) - \int_t^T Z^u(s)dW(s) + \int_t^T g(s,u,Y^u(s),Z^u(s))ds, \quad u \in \mathbb{R}^n, \tag{2.6}$$

where terminal condition and generator are subject to the following conditions

(C1) $g : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is an adapted measurable function such that $g(\omega,t,u,y,z) = l(\omega,t,u,y,z) + \alpha|z|^2$, where $l(\omega,t,u,y,z)$ is globally Lipschitz in

(y, z) and continuously differentiable in (u, y, z); for all $r \geq 1$ and (t, y, z) the mapping $\mathbb{R}^n \to L^r$, $u \mapsto l(\omega, t, u, y, z)$ is differentiable and for all $u \in \mathbb{R}^n$

$$\lim_{u' \to u} \mathbb{E}\left[\left(\int_0^T |l(s, u', Y^u(s), Z^u(s)) - l(s, u, Y^u(s), Z^u(s))|ds\right)^r\right] = 0 \quad \text{and} \quad (2.7)$$

$$\lim_{u' \to u} \mathbb{E}\left[\left(\int_0^T \left|\frac{\partial}{\partial u} l(s, u', Y^{u'}(s), Z^{u'}(s)) - \frac{\partial}{\partial u} l(s, u, Y^u(s), Z^u(s))\right| ds\right)^r\right] = 0 \qquad (2.8)$$

(C2) the random variables $\xi(u)$ are $\mathcal{F}_T$-adapted and for every compact set $K \subset \mathbb{R}^n$ there exists a constant $c \in \mathbb{R}$ such that $\sup_{u \in K} ||\xi(u)||_\infty < c$; for all $p \geq 1$ the mapping $\mathbb{R}^n \to L^p$, $u \mapsto \xi(u)$ is differentiable with derivative $\nabla\xi$.

If (C1) and (C2) are satisfied we know from Theorem 2.3 and Theorem 2.6 in Kobylanski (2000) that (2.6) has a unique solution. Furthermore the authors in Ankirchner et al. (2007) have derived under the conditions (C1) and (C2) that $u \mapsto (Y^u(\cdot), Z^u(\cdot))$ is differentiable, and the derivative is the unique solution to the BSDE

$$\begin{aligned}\nabla Y^u(t) = \nabla\xi - &\int_t^T \nabla Z^u(s) dW(s) \\ &+ \int_t^T [\partial_u l(s, u, Y^u(s), Z^u(s)) + \partial_y l(s, u, Y^u(s), Z^u(s))\nabla Y^u(s) \\ &+ \partial_z l(s, u, Y^u(s), Z^u(s))\nabla Z^u(s) + 2\alpha Z^u(s)\nabla Z^u(s)]ds\end{aligned} \qquad (2.9)$$

Compared to the BSDE (2.6) we consider less general BSDEs of type (2.4) in our approach to dynamic risk measures such that the conditions (C1) and (C2) simplify to a great extent. Note that the generator g in (2.4) does not depend on the process $Y(\cdot)$. In fact, it is property (D3) that requires the independence of the generator g of $Y(\cdot)$, see Barrieu and El Karoui (2009). Moreover, note that, in our case, condition (C2) is automatically satisfied, since we consider the function $\xi(u) = \xi + u\eta$ with $\xi, \eta \in L^\infty(\mathcal{F}_T)$. It follows immediately that there exists a constant $c \in \mathbb{R}$ such that

$$\sup_{u \in K} \|\xi(u)\|_\infty \leq \|\xi\|_\infty + \|\eta\|_\infty \sup_{u \in K} |u| < c$$

for every compact set $K \subset \mathbb{R}$. Furthermore the mapping $u \mapsto \xi(u)$ is differentiable with derivative $\nabla\xi = \eta$. From this discussion it follows that we consider parameter dependent BSDEs of type

$$Y^u(t) = -(\xi + u\eta) - \int_t^T Z^u(s) dW(s) + \int_t^T g(s, Z^u(s)) ds \qquad (2.10)$$

where terminal condition and generator are subject to the following condition

(A) $g : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is an adapted measurable function such that $g(\omega,t,z) = l(\omega,t,z) + \alpha|z|^2$, where $l(\omega,t,z)$ is globally Lipschitz in z and continuously differentiable in z.

Let $\xi = \sum_{i=1}^n \eta_i$. The result from Ankirchner et al. (2007) now enables us to define the BSDE-version of the dynamic gradient allocation as

$$\frac{d}{du}\rho(t;\xi + u\eta_i)\bigg|_{u=0} =: \nabla_{\eta_i} Y(t) \quad i = 1,\dots,n.$$

where we know that for each $i = 1,\dots,n$ under condition (A) $\nabla_{\eta_i} Y(t)$ exists as the first component of the solution to the BSDE

$$\nabla_{\eta_i} Y(t) = -\eta_i - \int_t^T \nabla_{\eta_i} Z(s)dW(s) + \int_t^T \partial_z g(s, Z^\xi(s))\nabla_{\eta_i} Z(s)ds. \tag{2.11}$$

These results enable us to provide in the next section our representation result for the dynamic gradient allocation.

3 Representation of Dynamic Capital Allocations

Theorem 3.1 *Let $\xi \in L^\infty(\mathcal{F}_T)$ and $\eta_i \in L^\infty(\mathcal{F}_T)$ for each $i = 1,\dots,n$. Suppose that $\xi = \sum_{i=1}^n \eta_i$ and let (A) hold. Suppose that $(\partial_z g(t, Z^\xi(t)))_{t\in[0,T]}$ is from $BMO(\mathbb{P})$ where $Z^\xi(\cdot)$ is the second component of the solution to BSDE* (2.4)*. Then the dynamic gradient allocation can be represented by*

$$\nabla_{\eta_i} Y(t) = \nabla_{\eta_i}\rho(t;\xi) = \mathbb{E}_{Q^\xi}[-\eta_i|\mathcal{F}_t], \quad i = 1,\dots,n,$$

where Q^ξ is given by

$$\frac{dQ^\xi}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} := \mathcal{E}\left(\int_0^\cdot \partial_z g(s, Z^\xi(s))dW(s)\right)(t) \tag{3.12}$$

Proof. Since $(\partial_z g(t, Z^\xi(t)))_{t\in[0,T]}$ is from $BMO(\mathbb{P})$ the Kazamaki Theorem (see Theorem 7.2 in Barrieu and El Karoui (2009)) allows us to define an equivalent probability measure as in (3.12). Since $W(\cdot)$ is a Brownian motion under $\mathbb{P}$, $\overline{W}(t) = W(t) - \int_0^t \partial_z g(s, Z^\xi(s))ds$ is a Brownian motion under Q^ξ. Define for each $i = 1,\dots,n$, $H_i(t) = \mathbb{E}_{Q^\xi}[-\eta_i|\mathcal{F}_t]$. Then from the martingale representation theorem it follows that there exists a process $Z^{\eta_i}(\cdot)$ such that

$$H_i(t) = H_i(T) - \int_t^T Z^{\eta_i}(s)d\overline{W}(s)$$

and the process $H_i(\cdot)$ satisfies

$$\begin{aligned} H_i(t) &= -\eta_i - \int_t^T Z^{\eta_i}(s)d\overline{W}(s) \\ &= -\eta_i - \int_t^T Z^{\eta_i}(s)dW(s) + \int_t^T Z^{\eta_i}(s)\partial_z g(s, Z^{\xi}(s))ds. \end{aligned} \tag{3.13}$$

If we compare (3.13) to the gradient allocation BSDE (2.11) and we know that under the assumptions of this theorem (2.11) has a unique solution we can conclude that the dynamic gradient allocation, which is the first component of the solution to (2.11), can be represented as

$$\nabla_{\eta_i} Y(t) = \nabla_{\eta_i}\rho(t;\xi) = \mathbb{E}_{Q^{\xi}}[-\eta_i|\mathcal{F}_t] \quad i = 1, \ldots, n.$$

with Q^{ξ} from (3.12). □

4 Representation of BSDE-based dynamic risk measures

The result from Theorem 3.1 immediately implies a representation result for BSDE-based dynamic convex and dynamic coherent risk measures. This result is based on the full allocation property of the Aumann-Shapley allocation which we have introduced in the static case in equation (2.2).

Corollary 4.1 *Let $\xi \in L^{\infty}(\mathcal{F}_T)$ and let (A) hold. Suppose that l is convex in z and $(\partial_z g(t, Z^{\gamma\xi}(t)))_{t\in[0,T]}$ is from $BMO(\mathbb{P})$ for any $\gamma \in [0,1]$ where $Z^{\gamma\xi}$, $\gamma \in [0,1]$, is the second component of the solution to BSDE (2.4) with terminal condition $Y^{\gamma\xi}(T) = -\gamma\xi$. Then the corresponding BSDE-based dynamic convex risk measure can be represented by*

$$\rho(t;\xi) = \mathbb{E}[-L^{\xi}(T,t)\xi|\mathcal{F}_t], \tag{4.14}$$

where $L^{\xi}(T,t)$ is given by

$$L^{\xi}(T,t) = \int_0^1 \frac{\mathcal{E}\left(\int_0^{\cdot} \partial_z g(s, Z^{\gamma\xi}(s))dW(s)\right)(T)}{\mathcal{E}\left(\int_0^{\cdot} \partial_z g(s, Z^{\gamma\xi}(s))dW(s)\right)(t)} d\gamma. \tag{4.15}$$

for any $t \in [0,T]$.

Proof. Based on the full allocation property of the Aumann-Shapley allocation, ie

$$\rho(t;\xi) = \int_0^1 \nabla_{\xi}\rho(t;\gamma\xi)d\gamma \tag{4.16}$$

which follows from

$$\begin{aligned} \rho(t;\xi) = \rho(t;1\xi) - \rho(t;0\xi) &= \int_0^1 \frac{d}{d\gamma}(\rho(t;\gamma\xi))d\gamma = \int_0^1 \lim_{\epsilon\to 0}\frac{\rho(t;(\gamma+\epsilon)\xi) - \rho(t;\gamma\xi)}{\epsilon}d\gamma \\ &= \int_0^1 \lim_{\epsilon\to 0}\frac{\rho(t;\gamma\xi+\epsilon\xi) - \rho(t;\gamma\xi)}{\epsilon}d\gamma = \int_0^1 \nabla_{\xi}\rho(t;\gamma\xi)d\gamma, \end{aligned}$$

Theorem 3.1 yields the representation

$$\rho(t;\xi) = \int_0^1 \mathbb{E}_{Q^{\gamma\xi}}[-\xi|\mathcal{F}_t]d\gamma. \tag{4.17}$$

Since $\xi \in L^\infty(\mathcal{F}_T)$ we know that ξ is $\mathbb{P}$-almost surely bounded. $Q^{\gamma\xi}$ is an equivalent probability measure for any $\gamma \in [0,1]$. Thus ξ is also $Q^{\gamma\xi}$-almost surely bounded, which means that there exists a constant $c \in \mathbb{R}_+$ such that $\mathbb{E}_{Q^{\gamma\xi}}[-\xi|\mathcal{F}_t] \le c$ $\mathbb{P}$-a.s. (and $Q^{\gamma\xi}$-a.s.) for any $\gamma \in [0,1]$. This implies

$$\int_0^1 \mathbb{E}_{Q^{\gamma\xi}}[-\xi|\mathcal{F}_t]d\gamma < \infty. \tag{4.18}$$

Denote by $L^{\gamma\xi}(t) = \mathcal{E}\left(\int_0^\cdot \partial_z g(s, Z^{\gamma\xi}(s))dW(s)\right)(t)$. Then, by interchanging the integrals in (4.17) we arrive at

$$\begin{aligned}\int_0^1 \mathbb{E}_{Q^{\gamma\xi}}[-\xi|\mathcal{F}_t]d\gamma &= \int_0^1 \frac{1}{L^{\gamma\xi}(t)}\mathbb{E}_{\mathbb{P}}[-L^{\gamma\xi}(T)\xi|\mathcal{F}_t]d\gamma = \mathbb{E}\left[-\left(\int_0^1 \frac{L^{\gamma\xi}(T)}{L^{\gamma\xi}(t)}d\gamma\right)\xi\,\middle|\,\mathcal{F}_t\right] \\ &= \mathbb{E}[-L^\xi(T,t)\xi|\mathcal{F}_t],\end{aligned}$$

which is the representation (4.14) with $L^\xi(T,t)$ from (4.15). □

Remark 4.2 *Note that the representation in Corollary 4.1 can also be obtained in the static case. Suppose that we have a (static) convex risk measure* $\rho : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ *with* $\rho(0) = 0$*. If* ρ *is Gâteaux-differentiable then the full allocation property of the Aumann-Shapley allocation holds. This means that we have*

$$\rho(X) = \int_0^1 \nabla_X \rho(\gamma X)d\gamma = \int_0^1 \mathbb{E}[\nabla\rho(\gamma X)X]d\gamma. \tag{4.19}$$

By interchanging the integrals in (4.19) *we arrive at a similar representation to* (4.14)*, namely*

$$\rho(X) = \mathbb{E}[L^X X],$$

where L^X *is given by*

$$L^X = \int_0^1 \nabla\rho(\gamma X)d\gamma.$$

If ρ *additionally satisfies the positive homogeneity property (S4) then* $\nabla\rho(\gamma X)$ *reduces to* $\nabla\rho(X)$ *and* L^X *becomes*

$$L^X = \int_0^1 \nabla\rho(X)d\gamma = \nabla\rho(X).$$

This is exactly the difference between the representation of convex and coherent risk measures. The representation of convex risk measures is based on a random variable that depends on all portfolio weights γ, $\gamma \in [0,1]$*, whereas the random variable in the representation of coherent risk measures depends only on the portfolio weight* $\gamma = 1$*. This means that size matters for convex risk measures. We will see in the following corollary that the same reasoning applies for BSDE-based dynamic coherent risk measures.*

Corollary 4.3 *Let $\xi \in L^\infty(\mathcal{F}_T)$ and let (A) hold. Suppose that g is of the form $g(t,z) = l(t,z)$ and l is convex and positively homogeneous in z. Furthermore suppose that $(\partial_z l(t, Z^{\gamma\xi}(t)))_{t\in[0,T]}$ is from $BMO(\mathbb{P})$. Then the corresponding BSDE-based dynamic coherent risk measure obtains the representation*

$$\rho(t;\xi) = \mathbb{E}_{Q^\xi}[-\xi|\mathcal{F}_t] \tag{4.20}$$

where the measure Q^ξ is given by

$$\left.\frac{dQ^\xi}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t \partial_z l(s, Z^\xi(s))dW(s) - \frac{1}{2}\int_0^t |\partial_z l(s, Z^\xi(s))|^2 ds\right\}. \tag{4.21}$$

Proof. We know from Corollary 4.1 that we have the following representation for $\rho(\cdot)$,

$$\rho(t;\xi) = \mathbb{E}[-L^\xi(T,t)\xi|\mathcal{F}_t],$$

with $L^\xi(T,t)$ from (4.15). Since g is of the form $g(t,z) = l(t,z)$ and l is convex and positively homogeneous this means that the corresponding BSDE-based dynamic risk measure $\rho(\cdot)$ satisfies

$$Y^{\gamma\xi}(t) = \rho(t;\gamma\xi) = \gamma\rho(t;\xi) = \gamma Y^\xi(t) \quad d\mathbb{P}\times dt\text{-a.s.} \tag{4.22}$$

for any $\gamma > 0$. To see this consider BSDE (2.4) and the BSDE

$$Y^{\gamma\xi}(t) = -\gamma\xi - \int_t^T Z^{\gamma\xi}(s)dW(s) + \int_t^T g(s, Z^{\gamma\xi}(s))ds. \tag{4.23}$$

Then, following the proof of Proposition 8 in Rosazza Gianin (2006), we can rewrite (4.23) as

$$\frac{Y^{\gamma\xi}(t)}{\gamma} = -\xi - \int_t^T \frac{Z^{\gamma\xi}(s)}{\gamma}dW(s) + \int_t^T g\left(s, \frac{Z^{\gamma\xi}(s)}{\gamma}\right) ds.$$

since g is assumed to be positively homogeneous in z and $\gamma > 0$. Then $\left(\frac{Y^{\gamma\xi}(\cdot)}{\gamma}, \frac{Z^{\gamma\xi}(\cdot)}{\gamma}\right)$ solves (2.4) and since the solution to (2.4) is unique we immediately get (4.22). Furthermore it follows that

$$Z^{\gamma\xi}(t) = \gamma Z^\xi(t) \quad d\mathbb{P}\times dt\text{-a.s.}$$

For the representation of BSDE-based coherent risk measures this means that the process $\mathcal{E}\left(\int_0^\cdot \partial_z g(s, Z^{\gamma\xi}(s))dW(s)\right)(\cdot)$ that appears in (4.15) becomes

$$\begin{aligned}
&\mathcal{E}\left(\int_0^\cdot \partial_z g(s, Z^{\gamma\xi}(s))dW(s)\right)(t)\\
&= \exp\left\{\int_0^t \partial_z g(s, \gamma Z^\xi(s))dW(s) - \frac{1}{2}\int_0^t |\partial_z g(s, \gamma Z^\xi)|^2 ds\right\}\\
&= \exp\left\{\int_0^t \partial_z g(s, Z^\xi(s))dW(s) - \frac{1}{2}\int_0^t |\partial_z g(s, Z^\xi)|^2 ds\right\}\\
&= \mathcal{E}\left(\int_0^\cdot \partial_z g(s, Z^\xi(s))dW(s)\right)(t)
\end{aligned}$$

because of the positive homogeneity of g in z. This means that $L^{\xi}(T,t)$, in the dynamic coherent case, is given by

$$L^{\xi}(T,t) = \int_0^1 \frac{\mathcal{E}\left(\int_0^{\cdot} \partial_z g(s, Z^{\xi}(s))dW(s)\right)(T)}{\mathcal{E}\left(\int_0^{\cdot} \partial_z g(s, Z^{\xi}(s))dW(s)\right)(t)} d\gamma = \frac{\mathcal{E}\left(\int_0^{\cdot} \partial_z g(s, Z^{\xi}(s))dW(s)\right)(T)}{\mathcal{E}\left(\int_0^{\cdot} \partial_z g(s, Z^{\xi}(s))dW(s)\right)(t)} \tag{4.24}$$

which leads to the representation (4.20) with Q^{ξ} given by (4.21). □

The result in Corollary 4.3 gives us a representation of BSDE-based dynamic coherent risk measures. For BSDE-based dynamic convex risk measures we have derived a representation in formula (4.14) where the representing exponential martingale, see (4.15), does not only depend on the portfolio ξ itself but on all scaled portfolios $\gamma\xi, \gamma \in [0,1]$. As already stated in Remark 4.2 this means that size matters for convex risk measures. This difference between the two representations is highlighted in (4.24) and it actually differentiates convex from coherent risk measures in an economic interpretation.

5 Examples

5.1 Dynamic and static entropic risk measure

In this subsection we will study the dynamic entropic risk measure, that is a representative of BSDE-based dynamic convex risk measures. The dynamic entropic risk measure is defined as

$$e_{\lambda}(t;\xi) = \lambda \ln \mathbb{E}\left[e^{-\frac{1}{\lambda}\xi} \middle| \mathcal{F}_t \right], \quad t \in [0,T].$$

We know from Proposition 6.4 in Barrieu and El Karoui (2009) that $(e_{\lambda}(t;\xi); t \in [0,T])$ is the first component of the solution to the BSDE

$$e_{\lambda}(t;\xi) = -\xi - \int_t^T Z^{\xi}(s)dW(s) + \int_t^T \frac{1}{2\lambda} Z^{\xi}(s)^2 ds.$$

Here the generator of the BSDE is given by $g(t,z) = \frac{1}{2\lambda}z^2$ such that we have

$$\partial_z g(s,z) = \frac{1}{\lambda} z.$$

Now, we are interested in the specific form of the process $Z^{\gamma\xi}(\cdot)$ that defines the representing exponential martingale of the entropic risk measure in (4.15).

Assume that ξ, from the class of smooth functionals $\mathcal{S}$, is a bounded random variable such that $(D_t^i\xi)_{t\in[0,T]}$, $i = 1, \ldots, d$, is from $BMO(\mathbb{P})$. Since this leads to

$$\|\xi\|_{1,1} := \mathbb{E}\left[|\xi| + \left(\sum_{i=1}^{d} \|D^i\xi\|^2 \right)^{1/2} \right] < \infty,$$

it follows that ξ is from $\mathbb{D}_{1,1}$. Define $\phi(x) = \exp(-(1/\lambda)x)$. Then from the boundedness of ξ and since ξ is from $\mathbb{D}_{1,1}$ it follows that

$$\mathbb{E}\left[|\phi(\xi)| + \left(\sum_{i=1}^{d} \|\phi'(\xi)D^i\xi\|^2\right)^{1/2}\right] < \infty,$$

such that we can apply the chain rule on $\mathbb{D}_{1,1}$ from Lemma A.1 in Ocone and Karatzas (1991) (see Theorem A.2 in Appendix A). This chain rule gives us $\phi(\xi) \in \mathbb{D}_{1,1}$ and $D\phi(\xi) = \phi'(\xi)D\xi = -(1/\lambda)\exp(-(1/\lambda)\xi)D\xi$. Now, for any $\gamma \in [0,1]$, it follows from the extended Clark-Ocone formula in Karatzas et al. (1991) (see Theorem A.1 in Appendix A) that

$$e^{-\frac{1}{\lambda}(\gamma\xi)} = \mathbb{E}[e^{-\frac{1}{\lambda}(\gamma\xi)}] + \int_0^T \mathbb{E}[D_t(e^{-\frac{1}{\lambda}(\gamma\xi)})|\mathcal{F}_t]dW(t).$$

Applying the conditional expectation and using the chain rule from above leads to

$$N^{\gamma\xi}(t) := \mathbb{E}[e^{-\frac{1}{\lambda}(\gamma\xi)}|\mathcal{F}_t] = \mathbb{E}[e^{-\frac{1}{\lambda}(\gamma\xi)}] + \int_0^t \mathbb{E}[-(\gamma/\lambda)e^{-\frac{\gamma}{\lambda}\xi}D_s\xi|\mathcal{F}_s]dW(s). \quad (5.25)$$

Since $N^{\gamma\xi}(\cdot)$ is a positive bounded martingale we can transform the above formula to

$$\begin{aligned} N^{\gamma\xi}(t) &= N^{\gamma\xi}(0) + \frac{1}{\lambda}\int_0^t N^{\gamma\xi}(s)\frac{-\gamma\mathbb{E}[e^{-\frac{\gamma}{\lambda}\xi}D_s\xi|\mathcal{F}_s]}{N^{\gamma\xi}(s)}dW(s) \\ &= N^{\gamma\xi}(0) + \frac{1}{\lambda}\int_0^t N^{\gamma\xi}(s)Z^{\gamma\xi}(s)dW(s), \end{aligned}$$

where the process $Z^{\gamma\xi}(\cdot)$ is defined by

$$Z^{\gamma\xi}(s) = \frac{-\gamma\mathbb{E}[e^{-\frac{\gamma}{\lambda}\xi}D_s\xi|\mathcal{F}_s]}{\mathbb{E}[e^{-\frac{\gamma}{\lambda}\xi}|\mathcal{F}_s]}. \quad (5.26)$$

From our assumptions on ξ we know that $Z^{\gamma\xi}(\cdot)$ from (5.26) is from $BMO(\mathbb{P})$. Moreover, $N^{\gamma\xi}(\cdot)$ satisfies

$$N^{\gamma\xi}(t) = N^{\gamma\xi}(0)\exp\left\{\frac{1}{\lambda}\int_0^t Z^{\gamma\xi}(s)dW(s) - \frac{1}{2\lambda^2}\int_0^t |Z^{\gamma\xi}(s)|^2 ds\right\}.$$

Now, the process $(N^{\gamma\xi}(\cdot)/N^{\gamma\xi}(0))$ corresponds to to the process $\mathcal{E}(X^{\gamma\xi})(\cdot)$ from (4.15) with $\partial_z g(s,z) = \frac{1}{\lambda}z$ and where the process $X^{\gamma\xi}(\cdot)$ is defined by

$$X^{\gamma\xi}(t) = \int_0^t \partial_z g(s, Z^{\gamma\xi}(s))dW(s), \quad t \in [0,T].$$

Thus we have identified the process $Z^{\gamma\xi}(\cdot)$ in (4.15) for the dynamic entropic risk measure by (5.26).

Let us now consider the static case. We have seen in Remark 4.2 that a Gâteaux-differentiable convex risk measure ρ with $\rho(0) = 0$ obtains the representation

$$\rho(X) = \mathbb{E}[L^X X],$$

where L^X is given by $L^X = \int_0^1 \nabla\rho(\gamma X)d\gamma$. In the following we will determine L^X for the (static) entropic risk measure, $e_\lambda : L^\infty \to \mathbb{R} \cup \{+\infty\}$,

$$e_\lambda(X) = \frac{1}{\lambda} \ln \mathbb{E}[\exp\{-\lambda X\}].$$

We know that the entropic risk measure is Gâteaux-differentiable. The Gâteaux-derivative of e_λ at $X \in L^\infty$ in the direction of $Y \in L^\infty$ is given by

$$\begin{aligned}\nabla_Y e_\lambda(X) &= -\frac{\mathbb{E}[Y \exp\{-\lambda X\}]}{\mathbb{E}[\exp\{-\lambda X\}]} \\ &= -\mathbb{E}\left[Y \frac{\exp\{-\lambda X\}}{\mathbb{E}[\exp\{-\lambda X\}]}\right]\end{aligned}$$

for any $Y \in L^\infty$. This yields

$$\nabla e_\lambda(\gamma X) = -\frac{\exp\{-\lambda\gamma X\}}{\mathbb{E}[\exp\{-\lambda\gamma X\}]}.$$

such that L^X is given by

$$L^X = \int_0^1 -\frac{\exp\{-\lambda\gamma X\}}{\mathbb{E}[\exp\{-\lambda\gamma X\}]} d\gamma$$

and e_λ can be represented by

$$e_\lambda(X) = \mathbb{E}\left[\int_0^1 -\frac{\exp\{-\lambda\gamma X\}}{\mathbb{E}[\exp\{-\lambda\gamma X\}]} d\gamma X\right]$$

5.2 Transformed norm risk measures

In Cheridito and Li (2008) the authors have introduced the class of transformed norm risk measures, that are defined by

$$\rho(X) = \min_{s \in \mathbb{R}} \left\{ \frac{1}{\alpha} \|(s - X)^+\|_p^\beta - s \right\} \tag{5.27}$$

with $(\alpha, \beta, p) \in (0, \infty) \times (1, \infty) \times [1, \infty)$ and where $\|\cdot\|_p^\beta$ is defined by $\|X\|_p^\beta = (\int |X|^p d\mathbb{P})^{\frac{\beta}{p}}$. For $\beta > 1$ the minimizer s_X of the right hand side of (5.27) is unique, see Cheridito and Li (2008). If $\beta, p > 1$ Proposition 3.2 and Proposition 7.1 in Cheridito and Li (2008) yield that ρ is Gâteaux-differentiable on L^p with

$$\nabla\rho(X) = \frac{((s_X - X)^+)^{p-1}}{\mathbb{E}[((s_X - X)^+)^{p-1}]},$$

see equation (8.6) in Cheridito and Li (2008). In this case L^X is given by

$$L^X = \int_0^1 \frac{((s_{\gamma X} - \gamma X)^+)^{p-1}}{\mathbb{E}[((s_{\gamma X} - \gamma X)^+)^{p-1}]} d\gamma$$

and ρ from (5.27) can be represented by

$$\rho(X) = \mathbb{E}\left[\left(\int_0^1 \frac{((s_{\gamma X} - \gamma X)^+)^{p-1}}{\mathbb{E}[((s_{\gamma X} - \gamma X)^+)^{p-1}]} d\gamma\right) X\right].$$

A Generalized Clark-Ocone formula and chain rule for Malliavin derivatives

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a standard, $\mathbb{R}^d$ valued Brownian motion $W(t) = (W_1(t), \ldots, W_d(t))^*$, $0 \le t \le T$ defined on it. Let $\{\mathcal{F}_t\}_{t \in [0,T]}$ denote the $\mathbb{P}$-augmentation of the natural Brownian filtration. Denote by $C_b^\infty(\mathbb{R}^m)$ the set of C^∞ functions $f : \mathbb{R}^m \to \mathbb{R}$ which are bounded and have bounded derivatives of all orders. Let $\mathcal{S}$ be the class of *smooth functionals*, i.e. random variables of the form

$$F(\omega) = f(W(t_1, \omega), \ldots, W(t_n, \omega)),$$

where $(t_1, \ldots, t_n) \in [0, T]^n$ and the function $f(x^{11}, \ldots, x^{d1}, \ldots, x^{1n}, \ldots, x^{dn})$ belongs to $C_b^\infty(\mathbb{R}^{dn})$. The gradient $DF(\omega)$ of the smooth functional F is defined as the $(L^2([0,T]))^d$-valued random variable $DF = (D^1F, \ldots, D^dF)$ with components

$$D^iF(\omega)(t) = \sum_{j=1}^n \frac{\partial}{\partial x^{ij}} f(W(t_1, \omega), \ldots, W(t_n, \omega)) 1_{[0,t_j]}(t), \quad i = 1, \ldots, d.$$

Let $|| \cdot ||$ denote the $L^2([0,T])$ norm and let $| \cdot |$ denote the Euclidean norm on $\mathbb{R}^n$. For each $p \ge 1$, we introduce the norm

$$\|F\|_{p,1} := \left(\mathbb{E}\left[|F|^p + \left(\sum_{i=1}^d \|D^iF\|^2\right)^{p/2}\right]\right)^{1/p}$$

on $\mathcal{S}$, and we denote by $\mathbb{D}_{p,1}$ the Banach space which is the closure of $\mathcal{S}$ under $|| \cdot ||_{p,1}$. Given $F \in \mathbb{D}_{p,1}$ we can find a measurable process $(t, \omega) \mapsto D_tF(\omega)$ such that for a.e. $\omega \in \Omega$, $D_tF(\omega) = DF(\omega)(t)$ holds for almost all $t \in [0, T]$. Now, we are able to state an extension of Clark's representation formula from Karatzas et al. (1991).

Theorem A.1 *For every $F \in \mathbb{D}_{1,1}$ we have*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[(D_tF)^*|\mathcal{F}_t] dW(t).$$

The following generalized chain rule is from Ocone and Karatzas (1991).

Theorem A.2 *Let* $F = (F_1, \ldots, F_k) \in (\mathbb{D}_{1,1})^k$. *Let* $\phi \in C^1(\mathbb{R}^k)$ *be a real-valued function and assume that*

$$\mathbb{E}\left[|\phi(F)| + \left\|\sum \frac{\partial \phi}{\partial x_i}(F) DF_i\right\|\right] < \infty.$$

Then $\phi(F) \in \mathbb{D}_{1,1}$ *and* $D\phi(F) = \sum(\partial\phi/\partial x_i)(F)DF^i$.

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