Differentiability of BSVIEs and Dynamic Capital Allocations

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Abstract

Capital allocations have been studied in conjunction with static risk measures in various papers. The dynamic case has been studied only in a discrete-time setting. We address the problem of allocating risk capital to subportfolios in a continuous-time dynamic context. For this purpose we introduce a differentiability result for backward stochastic Volterra integral equations and apply this result to derive continuous-time dynamic capital allocations. Moreover, we study a dynamic capital allocation principle that is based on backward stochastic differential equations and derive the dynamic gradient allocation for the dynamic entropic risk measure.

Keywords Dynamic risk capital allocation, dynamic risk measure, backward stochastic Volterra integral equation, backward stochastic differential equation, gradient allocation, Aumann-Shapley allocation.

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1 Introduction

Static coherent risk measures were introduced by Artzner et al. [3] and then generalized to static convex risk measures by Föllmer and Schied [13] and Frittelli and Rosazza Gianin [14]. These risk measures are static in the sense that they incorporate only the situation at a fixed terminal date. After the introduction of these types of risk measures several authors and the banking industry realized the importance of allocation techniques for risk-based portfolio management and risk-based performance measurement, see for instance Kalkbrener [19], McNeil et al. [25], Overbeck [26] and Tasche [31]. Capital allocations come into play if it is necessary to decompose

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a portfolio wide risk capital into a sum of risk contributions by sub-portfolios or single assets. Although in the static case many authors have approached the capital allocation problem from different starting points most of them identify the gradient allocation as an appropriate candidate for allocation schemes with coherent risk measures, see for instance [9], [10], [17], [21], [31]. For an axiomatic approach to capital allocations see Denault [10], Kalkbrener [17, 18]. If the underlying risk measure is not Gâteaux-differentiable the gradient allocation does not exist. If the reader is interested in capital allocation principles that do not require the underlying risk measure to have any specific properties we refer the reader to Cheridito and Kromer [7]. However in this work we will concentrate on the gradient allocation principle in conjunction with dynamic risk measures.

As stated in Jobert and Rogers [15] the usefulness of a single period study (static risk measures) should be judged by the extent to which it helps us to understand risk measurement in a multi-period setting. A multi-period structure requires, for instance, that information structure over time is taken into account. This has been recognized and dynamic risk measures were introduced and studied in various contexts in [4], [6], [11], [15], [28], [29], [30], [33] and others. The relevant approach to dynamic risk measures for this work is the construction of dynamic risk measures as solutions to backward stochastic differential equations (or BSDEs for short) or backward stochastic Volterra integral equations (or BSVIEs for short), which has been initiated by Peng [28] for BSDEs and by Yong [33] for BSVIEs. Compared to BSDE-based dynamic risk measures BSVIEs allow a more general study of dynamic risk measurement since position processes and not only $\mathscr{F}_T$-measurable random variables are incorporated into the approach.

BSDEs in general have been studied in Briand et al. [5], Kobylanski [20], Ma and Yong [23] and Pardoux and Peng [27]. Their important applications in finance appear for instance in Barrieu and El Karoui [4], El Karoui et al. [12], Mania and Schweizer [24] and Peng [28]. BSVIEs in general are studied in Anh et al. [1], Lin [22] and Yong [32, 34].

As static risk measures have been extended to dynamic risk measures we want to achieve in this paper the extension of capital allocations from the static to the continuous-time dynamic case. Thus the main contribution of this work is to establish the existence of the gradient allocation in the continuous-time dynamic risk setting. As noted before, we will hereby concentrate on a specific class of dynamic risk measures, namely the dynamic risk measures that arise as solutions to backward stochastic differential equations or to backward stochastic Volterra integral equations. To our knowledge there have been no investigations of allocation problems for continuous-time dynamic risk measures. Cherny [8] provides the only approach to dynamic capital allocations, that is known to us. This contribution nevertheless covers the discrete-time case, not the continuous-time version.

For that purpose we will provide a classical differentiability result for BSVIEs which is actually more general than needed for the direct application to the dynamic capital allocation problem. It will enable us to construct dynamic capital allocations as solutions to specific types of BSVIEs. This is the main contribution of the paper. Furthermore we will use an existing BSDE differentiability result from Ankirchner et al. [2] to construct the BSDE-version of the dynamic gradient allocation. The dynamic entropic risk measure and another dynamic risk measure that is based on BSVIEs are considered as examples for the derivation of the corresponding dynamic gradient allocation.

The outline of the paper is as follows. In Section 2 we present the necessary notation and provide the required definitions. We introduce convex and coherent risk measures with the corresponding gradient allocation principle in the static case. Furthermore we introduce BSVIEs and BSDEs. In Section 3 we discuss dynamic risk measures that arise from solutions to BSVIEs or BSDEs. In Section 4 we apply the differentiability result for BSVIEs to derive the BSVIE-version of the dynamic gradient allocation principle. Moreover we construct from an existing differentiability result for BSDEs the BSDE-based dynamic gradient allocation. In Section 5 we state and prove the general version of the classical differentiability result for BSVIEs.

2 Preliminaries

In this section we will collect definitions and notations that will be needed in the following sections. We will first introduce in short coherent and convex risk measures and then discuss backward stochastic differential equations and backward stochastic Volterra integral equations with the corresponding differences between these types of backward equations.

We will start with static convex and coherent risk measures. Consider $\mathscr{X}$ as some space of financial positions X that are defined on a probability space $(\Omega, \mathscr{G}, P)$. Here X is a random variable that represents a final payoff at some fixed time T. It could represent a cost generated by a firm or industry or simply a payoff of a portfolio. The map $\rho : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ is a (static) monetary risk measure if it satisfies

(S-TP) $\rho(X+r) = \rho(X) - r$ for any $X \in \mathscr{X}$ and any $r \in \mathbb{R}$,

(S-M) $\rho(X) \leq \rho(Y)$ for $X \geq Y$, $X,Y \in \mathscr{X}$.

It is a (static) convex risk measure if it satisfies in addition to (S-TP) and (S-M) the property

(S-C) $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, for $0 \leq \lambda \leq 1$ and any $X,Y \in \mathscr{X}$.

A (static) convex risk measure with the property

(S-PH) $\rho(\lambda X) = \lambda\rho(X)$ for any $X \in \mathscr{X}$ and $\lambda \in \mathbb{R}_+$,

is called (static) coherent risk measure. Coherent risk measures were introduced and studied in Artzner et al. [3]. Convex risk measures were introduced and studied in Föllmer and Schied [13] and Frittelli and Rosazza Gianin [14]. For further background literature on risk theory in the actuarial sciences we refer to Kaas et al. [16]. Static risk measures that are defined by the above properties do not allow for possible dynamic elements as additional information or intermediate payoffs. We will see in Section 3 that dynamic convex and dynamic coherent risk measures require additional properties that are concerned with the behavior of the risk measures through time. The "S" in the denomination of the properties above stands here for "static" to better distinguish between similar properties of dynamic risk measures that will be denoted with a "D" for "dynamic" instead.

Now suppose that $X \in \mathscr{X}$ represents a portfolio with risk $\rho(X)$ and $Y \in \mathscr{X}$ is a subportfolio of X. Then one of the basic problems of risk measurement is to determine the risk contribution of Y to X. If we have $\sum_{i=1}^{n} X_i = X$ then the objective is to compute the risk contributions of each X_i, $i = 1, \ldots, n$ to X, the risk capital has to be allocated among its constituent parts. This is then called risk capital allocation. As already stated in the introduction, in the static case the gradient allocation has proved to be an appropriate candidate for allocation schemes with coherent risk measures, see for instance Cherny and Orlov [9], Denault [10], Kalkbrener [17], Kromer and Overbeck [21] and Tasche [31]. For a Gâteaux-differentiable risk measure ρ the (static) gradient allocation principle is defined as

$$\rho(X;X_i) := \lim_{u\to 0} \frac{\rho(X+uX_i) - \rho(X)}{u} = \frac{d}{du}\rho(X+uX_i)\bigg|_{u=0} \tag{2.1}$$

Our aim in this paper is to establish the existence of the gradient allocation in the continuous-time dynamic risk setting, where dynamic risk measures are constructed from solutions to BSDEs or BSVIEs. On that account we need to introduce BSDEs and BSVIEs.

Denote by $\mathbb{R}^m$, $m \in \mathbb{N}$, the m-dimensional Euclidean space with the usual L^2-norm, denoted by $|\,.\,|$. Let $\mathbb{R}^{m\times d}$ be the Hilbert space of all $m \times d$ matrices with the inner product defined by

$$\langle A, B\rangle = \mathrm{tr}[AB^T],$$

for all $A, B \in \mathbb{R}^{m\times d}$.

Let from now on for the rest of the paper $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a d-dimensional Brownian motion $(W(t))_{t\geq 0}$ is defined and $\mathbb{F} = \{\mathscr{F}_t\}_{t\geq 0}$ is its natural filtration augmented by the $\mathbb{P}$-null sets in $\mathscr{F}$. Consider the following stochastic integral equation

$$Y(t) = \psi(t) - \int_t^T Z(t,s)dW(s) + \int_t^T g(t,s,Y(s),Z(t,s),Z(s,t))ds, \quad t \in [0,T], \tag{2.2}$$

where $g : \Delta^c \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d} \times \Omega \to \mathbb{R}^m$ and $\psi : [0,T] \times \Omega \to \mathbb{R}^m$ are given maps and the sets Δ and Δ^c are defined as

$$\Delta := \{(t,s) \in [0,T]^2 : t \geq s\} \quad \text{and} \quad \Delta^c := \{(t,s) \in [0,T]^2 : t < s\}.$$

The stochastic integral equation in (2.2) is called backward stochastic Volterra integral equation. An adapted solution to (2.2) is the pair $(Y(\cdot), Z(\cdot,\cdot))$, where $Y(\cdot)$ is an $\mathbb{F}$-adapted process and $Z(t,\cdot)$ is $\mathbb{F}$-adapted for almost all $t \in [0,T]$.

A special case of the above equation is the following:

$$Y(t) = \xi - \int_t^T Z(s)dW(s) + \int_t^T g(s,Y(s),Z(s))ds, \quad t \in [0,T], \tag{2.3}$$

where $g : [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$ and ξ is an $\mathscr{F}_T$-measurable L^p-integrable random variable for some $p > 1$. The differential form of (2.3) is called backward stochastic differential equation (or BSDE for short). For detailed discussions on BSDEs we refer the reader to Briand et al. [5], Kobylanski [20], Ma and Yong [23] and Pardoux and Peng [27].

We immediately see that BSVIE (2.2) is a natural generalization to BSDE (2.3). As already noted in Yong [34] the main differences of BSVIE (2.2) to BSDE (2.3) are:

(i) the generator g depends on both t and s, which implies that the equation cannot be reduced to a BSDE in general

(ii) the generator g depends not only on $Z(t,s)$, but also on $Z(s,t)$ (this will be important later on in Section 3)

(iii) the process $\psi(\cdot)$ is allowed to be just $\mathscr{B}[0,T] \otimes \mathscr{F}_T$-measurable (not necessarily $\mathbb{F}$-adapted), where $\mathscr{B}[0,T]$ is the Borel σ-field of $[0,T]$.

Now let $H = \mathbb{R}^m, \mathbb{R}^{m\times d}$ and $S \in [0,T]$. To study BSVIEs of type (2.2) we will introduce the following spaces.

$$L^2_{\mathscr{F}_S}(0,T) = \left\{ \varphi(\cdot) : (0,T) \times \Omega \to H \,\middle|\, \varphi(\cdot) \text{ is } \mathscr{B}([0,T]) \otimes \mathscr{F}_S\text{-measurable and } \mathbb{E}\left[\int_0^T |\varphi(t)|^2 dt\right] < \infty \right\} \tag{2.4}$$

If $S = T$ we will suppress the subscript $\mathscr{F}_S$ and write $L^2(0,T)$. In the above definition $\mathbb{F}$-adaptiveness is not involved and for the process $Y(\cdot)$ (where $\mathbb{F}$-adaptiveness is involved) we will need the following space.

$$L^2_{\mathbb{F}}(0,T) = \{\varphi(\cdot) \in L^2(0,T) \,|\, \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted}\} \tag{2.5}$$

For the process $Z(\cdot,\cdot)$ we introduce the space $L^2(0,T;L^2_{\mathbb{F}}(0,T))$, which is the space of all processes $Z : [0,T]^2 \times \Omega \to H$ such that for almost all $t \in [0,T]$ $Z(t,\cdot) \in L^2_{\mathbb{F}}(0,T)$ and

$$\mathbb{E}\left[\int_0^T \int_0^T |Z(t,s)|^2 ds dt\right] < \infty. \tag{2.6}$$

In all the definitions of the spaces above $[0,T]$ can be replaced by any $[R,S]$ with $0 \le R < S \le T$. For solutions to BSVIEs of type (2.2) we will need the space

$$\mathscr{H}^2[R,S] = L^2_{\mathbb{F}}(R,S) \times L^2(R,S;L^2_{\mathbb{F}}(R,S)), \tag{2.7}$$

which is a Banach space with norm

$$||Y(\cdot),Z(\cdot,\cdot)||^2_{\mathscr{H}^2[R,S]} \equiv \mathbb{E}\left[\int_R^S |Y(t)|^2 dt + \int_R^S \int_R^S |Z(t,s)|^2 ds dt\right]. \tag{2.8}$$

From Example 1.1 in [34] we know that an adapted solution to BSVIE (2.2) (in the usual Itô-sense) cannot be unique. For the precise reason for that see [34], Section 3. Therefore, in order to have uniqueness we need additional conditions imposed on $Z(t,s)$ for $(t,s) \in \Delta$. This leads to the introduction of the M-solution.

Definition 2.1. *Let $S \in [0,T)$. A pair $(Y(\cdot),Z(\cdot,\cdot)) \in \mathscr{H}^2[S,T]$ is called an adapted M-solution to BSVIE* (2.2) *on $[S,T]$ if* (2.2) *holds in the usual Itô-sense for almost all $t \in [S,T]$ and, in addition it holds*

$$Y(t) = \mathbb{E}[Y(t)|\mathscr{F}_S] + \int_S^t Z(t,s) dW(s), \quad \textit{for a.e. } t \in [S,T]. \tag{2.9}$$

"M-solution" stands here for "martingale representation" and the additional constraint (2.9) will guarantee uniqueness of solutions.

Before we move on to the next section and the introduction of dynamic risk measures that are based on BSVIEs we will state an existence and uniqueness result for M-solutions to BSVIEs of type (2.2) from Yong [34]. We will use this important result frequently in the ongoing discussion. Let us first state the needed conditions on the driver of the BSVIE.

(H1) Let $g : \Delta^c \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d} \times \Omega \to \mathbb{R}^m$ be $\mathscr{B}(\Delta^c \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d}) \otimes \mathscr{F}_T$-measurable such that $s \mapsto g(t,s,y,z,\zeta)$ is $\mathbb{F}$-progressively measurable for all $(t,y,z,\zeta) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d}$ and

$$\mathbb{E}\left[\int_0^T \left(\int_0^T |g(t,s,0,0,0)| ds\right)^2 dt\right] < \infty. \tag{2.10}$$

Furthermore it holds

$$|g(t,s,y,z,\zeta) - g(t,s,\overline{y},\overline{z},\overline{\zeta})| \le L(t,s)(|y-\overline{y}| + |z-\overline{z}| + |\zeta - \overline{\zeta}|),$$

for all $(t,s) \in \Delta^c$, $y,\overline{y} \in \mathbb{R}^m$, $z,\overline{z},\zeta,\overline{\zeta} \in \mathbb{R}^{m\times d}$ a.s., where $L : \Delta^c \to [0,\infty)$ is a deterministic function such that the following holds:

$$\sup_{t\in[0,T]} \int_t^T L(t,s)^{2+\varepsilon} ds < \infty \tag{2.11}$$

for some $\varepsilon > 0$.

With condition (H1) we are now able to state the following result from Yong [34].

Theorem 2.2. *Let (H1) hold. Then for any $\psi(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$, BSVIE (2.2) admits a unique adapted M-solution $(Y(\cdot), Z(\cdot,\cdot)) \in \mathscr{H}^2[0,T]$ on $[0,T]$. Moreover, the following estimate holds:*

$$\begin{aligned}||(Y(\cdot),Z(\cdot,\cdot))||^2_{\mathscr{H}^2[S,T]} &\equiv \mathbb{E}\left[\int_S^T |Y(t)|^2 dt + \int_S^T \int_S^T |Z(t,s)|^2 ds dt\right] \\ &\leq C\mathbb{E}\left[\int_S^T |\psi(t)|^2 dt + \int_S^T \left(\int_t^T |g_0(t,s)| ds\right)^2 dt\right] \end{aligned} \tag{2.12}$$

for all $S \in [0,T]$.

Now, we can proceed to the next section and introduce dynamic risk measures that are based on BSVIEs or BSDEs that are special cases of BSVIE (2.2), respectively BSDE (2.3).

3 Dynamic Risk Measures based on BSVIEs or BSDEs

Yong [33] introduced the notion of dynamic risk measures that are based on BSVIEs of the following type

$$Y(t) = -\psi(t) + \int_t^T g(t,s,Y(s),Z(s,t))ds - \int_t^T Z(t,s)dW(s), \quad t \in [0,T], \tag{3.13}$$

where $g : \Delta^c \times \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a given map. This approach to dynamic risk measures was inspired by the previous study of dynamic risk measures that are based on backward stochastic differential equations from El Karoui et al. [12], Peng [28] and Rosazza-Gianin [30]. We will study the BSDE-approach and highlight the main differences to the BSVIE-approach later on.

In this section we will first present the results on dynamic risk measures from Yong [33]. We will start with the definition of dynamic risk measures based on BSVIEs. Note that we will use for simplicity of notation the condition (H1) from Section 2 although we would only need a simplified version of it since we study here only the one-dimensional case and g does not depend on $Z(t,s)$ as in BSVIE (2.2).

Definition 3.1. *A map $\rho : L^2_{\mathscr{F}_T}(0,T) \to L^2_{\mathbb{F}}(0,T)$ is called BSVIE-based dynamic risk measure if the following holds:*

*(i) **(Past Independence: (D-PI))** For any $\psi(\cdot), \overline{\psi}(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$, if*

$$\psi(s) = \overline{\psi}(s) \quad a.s.,\ s \in [t,T]$$

for some $t \in [0,T)$, then

$$\rho(t;\psi(\cdot)) = \rho(t;\overline{\psi}(\cdot)) \quad a.s.$$

*(ii) **(Monotonicity: (D-M))** For any $\psi(\cdot), \overline{\psi}(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$, if*

$$\psi(s) \leq \overline{\psi}(s) \quad a.s.,\ s \in [t,T]$$

for some $t \in [0,T)$, then

$$\rho(s;\psi(\cdot)) \geq \rho(s;\overline{\psi}(\cdot)) \quad a.s.,\ s \in [t,T].$$

One should note that for any position process $\psi(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$, $t \mapsto \rho(t;\psi(\cdot))$ is $\mathbb{F}$-adapted.

Past independence in Definition 3.1 means that the dynamic risk measure should only depend on the current and future information of the position process $\psi(\cdot)$.

Now let us introduce additional properties a dynamic risk measure can have. These properties are similar to the properties of risk measures in the static case (see Section 2) and will lead to the definitions of dynamic convex and dynamic coherent risk measures based on BSVIEs.

(Convexity: (D-C)) For any $\psi(\cdot), \overline{\psi}(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$ and $\lambda \in [0,1]$

$$\rho(t;\lambda\psi(\cdot)+(1-\lambda)\overline{\psi}(\cdot)) \leq \lambda\rho(t;\psi(\cdot))+(1-\lambda)\rho(t;\overline{\psi}(\cdot)) \quad \text{a.s.}$$

(Translation Property: (D-TP)) There exists a deterministic integrable function $r(\cdot)$ such that for any $\psi(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$

$$\rho(t;\psi(\cdot)+c) = \rho(t;\psi(\cdot)) - ce^{-\int_t^T r(s)ds} \quad \text{a.s., } t \in [0,T].$$

(Positive Homogeneity: (D-PH)) For any $\psi(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$ and $\lambda > 0$

$$\rho(t;\lambda\psi(\cdot)) = \lambda\rho(t;\psi(\cdot)) \quad \text{a.s., } t \in [0,T].$$

(Subadditivity: (D-S)) For any $\psi(\cdot), \overline{\psi}(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$

$$\rho(t;\psi(\cdot)+\overline{\psi}(\cdot)) \leq \rho(t;\psi(\cdot))+\rho(t;\overline{\psi}(\cdot)) \quad \text{a.s., } t \in [0,T].$$

Note that, as in the static case, (D-PH) and (D-S) imply (D-C). With the properties above we are now able to define BSVIE-based dynamic convex risk measures and BSVIE-based dynamic coherent risk measures as follows.

Definition 3.2. *(i) A BSVIE-based dynamic risk measure $\rho : L^2_{\mathscr{F}_T}(0,T) \to L^2_{\mathbb{F}}(0,T)$ is called BSVIE-based dynamic convex risk measure if it satisfies (D-C) and (D-TP).*

(ii) A BSVIE-based dynamic risk measure $\rho : L^2_{\mathscr{F}_T}(0,T) \to L^2_{\mathbb{F}}(0,T)$ is called BSVIE-based dynamic coherent risk measure if it satisfies (D-PH), (D-S) and (D-TP).

The question whether such dynamic risk measures as in Definition 3.1 and Definition 3.2 can be constructed for general position processes $\psi(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$ is positively answered in Yong [33]. Consider the BSVIE (3.13). The adapted solution to this BSVIE, which is a special case of BSVIE (2.2), was used in [33] to identify dynamic risk measures. One only has to suitably choose the generator g for BSVIE (3.13) to easily construct a large class of dynamic convex or dynamic coherent risk measures that are based on BSVIEs.

Define

$$\rho(t;\psi(\cdot)) = Y(t) \quad \text{for all } t \in [0,T], \tag{3.14}$$

where $(Y(\cdot), Z(\cdot,\cdot))$ is the unique adapted M-solution to BSVIE (3.13). Then [33] provides the following result.

Theorem 3.3. *Let $\psi(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$. Suppose that $g(\cdot)$ satisfies (H1) and $g(\cdot)$ is the form*

$$g(t,s,y,\zeta) = r(s)y + g_0(t,s,\zeta), \tag{3.15}$$

with $r(\cdot)$ being a bounded deterministic function. Then $\rho(\cdot)$ defined by (3.14)*, where $Y(\cdot)$ is the unique adapted M-solution to the BSVIE* (3.13)*, is a dynamic convex risk measure if $\zeta \mapsto g_0(t,s,\zeta)$ is convex, namely $g_0(\cdot)$ satisfies*

$$g_0(t,s,\lambda\zeta_1+(1-\lambda)\zeta_2) \leq \lambda g_0(t,s,\zeta_1)+(1-\lambda)g_0(t,s,\zeta_2),$$

for any $\zeta_1,\zeta_2 \in \mathbb{R}^d$, $\lambda \in [0,1]$ and $(t,s) \in [0,T]^2$ a.s. It is a dynamic coherent risk measure if $\zeta \mapsto g_0(t,s,\zeta)$ is positively homogeneous and subadditive, i.e. it satisfies

$$\begin{aligned} g_0(t,s,\lambda\zeta) &= \lambda g_0(t,s,\zeta) \quad \text{and} \\ g_0(t,s,\zeta_1+\zeta_2) &\leq g_0(t,s,\zeta_1)+g_0(t,s,\zeta_2), \end{aligned}$$

for any $(t,s) \in [0,T]^2$, $\zeta,\zeta_1,\zeta_2 \in \mathbb{R}^d$ and $\lambda > 0$ a.s.

We will study an example of a BSVIE-based dynamic convex risk measure in the next section, see Example 4.4.

Note that if $r(\cdot)=0$ in (3.15), the driver g in (3.13) is independent of the process $Y(\cdot)$ and the translation property (D-TP) becomes

$$\rho(t;\psi(\cdot)+c) = \rho(t;\psi(\cdot))-c,$$

which is similar to the classical one from the static case, see (S-TP). With this the following result holds.

Corollary 3.4. *Let $\psi(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$. Suppose that $g_0(\cdot)$ satisfies (H1). Then $\rho(\cdot)$ defined by* (3.14)*, where $Y(\cdot)$ is the first component of the unique adapted M-solution to the BSVIE*

$$Y(t) = -\psi(t)+\int_t^T g_0(t,s,Z(s,t))ds - \int_t^T Z(t,s)dW(s), \quad t \in [0,T],$$

is a dynamic convex risk measure if $\zeta \mapsto g_0(t,s,\zeta)$ is convex. It is a dynamic coherent risk measure if $\zeta \mapsto g_0(t,s,\zeta)$ is positively homogeneous and subadditive.

From the previous results (Theorem 3.3 and Corollary 3.4) we see that the properties of the generator g of the corresponding BSVIE translate to the properties of the BSVIE-based dynamic risk measure. This is known to be just the case for BSDE-based dynamic risk measures as we will see in the following.

Let us first state a few important results on BSDEs. Consider the following stochastic integral equation

$$Y(t) = \xi + \int_t^T g(s,Y(s),Z(s))ds - \int_t^T Z(s)dW(s), \tag{3.16}$$

where $g:[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\Omega \to \mathbb{R}$ is a given function and ξ is an $\mathscr{F}_T$-measurable random variable. The differential version of this equation is usually called a backward stochastic differential equation. A solution consists of a pair $(Y(\cdot),Z(\cdot))$ of adapted processes such that (3.16) is satisfied. Now, suppose that the generator satisfies for $a \geq 0$ and $b,c > 0$

$$|g(t,y,z)| \leq a(1+b|y|)+\frac{c}{2}|z|^2. \tag{3.17}$$

With this assumption on the generator of the BSDE (3.16) Kobylanski [20] showed that if ξ is bounded, then there exists a solution $(Y,Z) \in \mathscr{R}^\infty \times \mathbb{L}^2$, where $\mathscr{R}^\infty$ is the space of bounded measurable processes and $\mathbb{L}^2(0,T)$ is the space of all progressively measurable processes $X(\cdot)$ such that

$$\mathbb{E}\left[\int_0^T |X(s)|^2 ds\right] < \infty.$$

Note that, to our knowledge, such an existence result for BSVIEs with a quadratic driver does not exist. For that reason we work only with Lipschitz-continuous drivers of BSVIEs and are more flexible with the drivers of BSDEs. The quadratic driver of the BSDE will allow us to study the dynamic entropic risk measure in this context (see Example 4.5).

Let us first specify the properties that we will need for the definition of BSDE-based dynamic risk measures. Several of them are similar to the properties of the BSVIE-based approach, but there are also differences. Instead of the past independence property (D-PI) that was used in the definition of BSVIE-based dynamic risk measures we will need the time consistency and the arbitrage freedom property for the definition of BSDE-based dynamic risk measures. These are defined as follows.

(Time Consistency: (D-TC)) For three bounded stopping times $S \le T \le U$ and any bounded $\mathscr{F}_U$-measurable random variable ξ^U

$$\rho(S;\xi^U) = \rho(S; -\rho(T;\xi^U)) \quad \text{a.s.}$$

(Arbitrage freedom: (D-AF)) For any stopping times $S \le T$ and any bounded $\mathscr{F}_T$-measurable random variables ξ_1^T and ξ_2^T such that $\xi_1^T \ge \xi_2^T$ a.s.

$$\rho(S;\xi_1^T) = \rho(S;\xi_2^T) \quad \text{on} \quad A_S = \{S < T\} \quad \Longrightarrow \quad \xi_1^T = \xi_2^T \quad \text{a.s. on } A_S.$$

Now, following Definition 6.2 in [4] we can define a BSDE-based dynamic convex risk measure in this context as a map from $L^\infty(\mathscr{F}_T)$ to $\mathscr{R}^\infty(0,T)$ that satisfies (D-TC) and (D-AF) and the analogous properties to (D-M), (D-C) and (D-TP) with $r(\cdot) = 0$. For the corresponding properties of BSDE-based dynamic risk measures one only has to replace the position process $\psi(\cdot)$ by the $\mathscr{F}_T$-measurable random variable ξ such that we won't repeat these properties here.

Similar to the BSVIE-approach we can construct BSDE-based dynamic risk measures from

$$\rho(t;\xi) = Y(t) \quad \text{for all } t \in [0,T], \tag{3.18}$$

where $(Y(\cdot), Z(\cdot))$ is the unique adapted solution to the BSDE

$$Y(t) = -\xi - \int_t^T Z(s)dW(s) + \int_t^T g(s, Z(s))ds, \tag{3.19}$$

where g is of the form

$$g(t,z) = l(t,z) + \alpha |z|^2, \tag{3.20}$$

$(l(t,0); 0 \le t \le T)$ is progressively measurable, satisfies

$$\mathbb{E}\left[\int_0^T |l(s,0)|^2 ds\right] < \infty$$

and l is Lipschitz continuous in z. The independence of the driver g of y is based on the definition of the translation property in [4], where in comparison to (D-TP) the function $r(\cdot)$ is assumed to be zero (see Corollary 3.4). If l is convex in z then Theorem 6.7 from Barrieu and El Karoui [4] tells us that $\rho(\cdot)$ from (3.18) is a BSDE-based dynamic convex risk measure. If $g(t,z) = l(t,z)$ in (3.20) and l is convex and positively homogeneous in z then $\rho(\cdot)$ from (3.18) is a BSDE-based dynamic coherent risk measure.

Let us now proceed to the next section where we will study dynamic capital allocations that are based on BSVIEs or BSDEs.

4 Dynamic Capital Allocations

Let us start this section with the differentiability result for specific types of BSVIEs. We will not prove this result here, because it follows from a much more general differentiability result in the next section. This result here will nevertheless allow us to identify the BSVIE-based dynamic gradient allocation from solutions to linear BSVIEs.

Consider the following parameter dependent BSVIEs

$$Y^u(t) = -(\psi(t) + u\varphi(t)) + \int_t^T g(t,s,Y^u(s),Z^u(s,t))ds - \int_t^T Z^u(t,s)dW(s), \tag{4.21}$$

for any $u \in \mathbb{R}$.

Theorem 4.1. *Let (D1) from Section 5 hold. Then the function*

$$\mathbb{R} \to L^2_{\mathbb{F}}(0,T) \times L^2_{\mathbb{F}}(0,T; L^2_{\mathbb{F}}(0,T)), \quad u \mapsto (Y^u(\cdot), Z^u(\cdot,\cdot))$$

is differentiable and the derivative is the unique adapted M-solution to the BSVIE

$$\begin{aligned}\nabla Y^u(t) = -\varphi(t) &- \int_t^T \nabla Z^u(t,s)dW(s)\\ &+ \int_t^T \Big[\partial_y g(t,s,Y^u(s),Z^u(s,t))\nabla Y^u(s)\\ &\qquad + \sum_{j=1}^d \partial_{\zeta_j} g(t,s,Y^u(s),Z^u(s,t))\nabla Z^u_j(s,t)\Big]ds\end{aligned} \tag{4.22}$$

Proof. This result follows directly from the more general result in Theorem 5.2. □

Now, with this differentiability result we can identify the dynamic gradient allocation with solutions to specific types of BSVIEs. Define the dynamic gradient allocation with respect to the BSVIE-based dynamic risk measure from (3.14) as

$$\frac{d}{du}\rho(t;\psi(\cdot)+u\varphi(\cdot))\Big|_{u=0} = \nabla Y^u(t)|_{u=0} =: \nabla Y(t), \tag{4.23}$$

where $(\nabla Y^u(\cdot), \nabla Z^u(\cdot,\cdot))$, if it exists, is the adapted M-solution to BSVIE (4.22).

Consider $\varphi(\cdot)$ to be a subportfolio process of the portfolio process $\psi(\cdot)$. Then (4.23) describes the time t-dependent risk capital the subportfolio process $\varphi(\cdot)$ contributes to the portfolio position process $\psi(\cdot)$. This is the continuous-time dynamic risk capital that should be allocated to $\varphi(\cdot)$.

In the next theorem, which is the main result of this section, we will show that such a dynamic capital allocation exists and can be constructed as a solution to a BSVIE, as suggested in (4.23).

Theorem 4.2. *Let (D1) hold. Let $\psi(\cdot), \varphi(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$. Suppose that* (3.15) *holds for some bounded and deterministic function $r(\cdot)$. Then $\rho(\cdot)$ defined by* (3.14) *where $(Y(\cdot), Z(\cdot,\cdot))$ is the unique adapted M-solution to BSVIE* (3.13) *is a dynamic risk measure. Moreover the corresponding dynamic gradient allocation defined by* (4.23) *exists as the unique adapted M-solution to BSVIE*

$$\begin{aligned}\nabla Y(t) = -\varphi(t) &- \int_t^T \nabla Z(t,s)dW(s)\\ &+ \int_t^T \Big[r(s)\nabla Y(s) + \sum_{j=1}^d \partial_{\zeta_j} g_0(t,s,Z(s,t))\nabla Z_j(s,t)\Big]ds\end{aligned} \tag{4.24}$$

Proof. Since condition (D1) from Section 5 is stronger than (H1) it follows directly from Theorem 3.3 that $\rho(\cdot)$ is a dynamic risk measure. Consider now the BSVIE (4.21). Then, since g satisfies (D1), (4.21) has a unique adapted M-solution $(Y^u(\cdot), Z^u(\cdot,\cdot))$ and

$$\rho(t;\psi(\cdot)+u\varphi(\cdot)) = Y^u(t) \tag{4.25}$$

defines a dynamic risk measure at position process $(\psi(\cdot)+u\varphi(\cdot)) \in L^2_{\mathscr{F}_T}(0,T)$ for any $u \in \mathbb{R}$. Now from Theorem 4.1 it follows that $(\nabla Y(\cdot), \nabla Z(\cdot,\cdot))$ with $\frac{d}{du}\rho(t;\psi(\cdot)+u\varphi(\cdot))|_{u=0} = \nabla Y(t)$ is the unique adapted M-solution to BSVIE (4.24) such that the dynamic gradient allocation

$$\frac{d}{du}\rho(t;\psi(\cdot)+u\varphi(\cdot))\bigg|_{u=0}$$

exists as the first component of the unique adapted M-solution to (4.24). □

If $r(\cdot) = 0$ in (3.15), the driver g in (4.21) is independent of the process $Y(\cdot)$ and we get the following Corollary.

Corollary 4.3. *Let (D1) hold. Let $\psi(\cdot), \varphi(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$. Then $\rho(\cdot)$, defined by* (3.14)*, where $(Y(\cdot), Z(\cdot,\cdot))$ is the unique adapted M-solution to BSVIE*

$$Y(t) = -\psi(t) - \int_t^T Z(t,s)dW(s) + \int_t^T g(t,s,Z(s,t))ds, \quad t \in [0,T], \tag{4.26}$$

is a dynamic risk measure. Moreover, the corresponding dynamic gradient allocation defined by (4.23) *exists, where $\nabla Y(t) = \frac{d}{du}\rho(t;\psi(\cdot)+u\varphi(\cdot))|_{u=0}$ is the first component of the unique adapted M-solution to the BSVIE*

$$\nabla Y(t) = -\varphi(t) - \int_t^T \nabla Z(t,s)dW(s) + \int_t^T \sum_{j=1}^d [\partial_{\zeta_j} g(t,s,Z(s,t))\nabla Z_j(s,t)]ds. \tag{4.27}$$

Example 4.4. *[33] studies the following example of a dynamic convex risk measure. Consider the following BSVIE:*

$$Y(t) = -\psi(t) - \int_t^T Z(t,s)dW(s) + \int_t^T (r(s)Y(s) + g_0(t,s)\sqrt{1+|Z(s,t)|^2})ds, \tag{4.28}$$

with a properly selected function $g_0(t,s)$ and with a bounded deterministic function $r(\cdot)$. The generator in BSVIE (4.28) *is given by*

$$g(t,s,y,\zeta) = r(s)y + g_0(t,s)\sqrt{1+|\zeta|^2}. \tag{4.29}$$

It satisfies conditions (H1) and (3.15) *and it is convex in ζ. Therefore, Theorem 3.3 implies that $\rho(\cdot)$, which is defined as the first component of the M-solution to* (4.28)*, is a dynamic convex risk measure for the position process $\psi(\cdot)$. As noted in Yong [33], the choice of $g_0(\cdot,\cdot)$ represents the agent's attitude towards risk; the larger $g_0(\cdot,\cdot)$, the more risk averse is the agent.*

We will derive for this specific dynamic convex risk measure the corresponding dynamic gradient allocation. Since the generator g, given in (4.29) *satisfies (D1) we can apply Theorem 4.2. It follows that the dynamic gradient allocation with respect to the dynamic convex risk measure from*

(4.28) *is given for any* $\psi(\cdot),\varphi(\cdot)\in L^2_{\mathscr{F}_T}(0,T)$ *from* (4.23)*, where* $\nabla Y(\cdot)$ *is the first component of the M-solution to the following BSVIE*

$$\nabla Y(t) = -\varphi(t) - \int_t^T \nabla Z(t,s)dW(s) + \int_t^T \left[r(s)\nabla Y(s) + g_0(t,s)\sum_{j=1}^d \frac{Z_j(s,t)}{\sqrt{1+|Z(s,t)|^2}}\nabla Z_j(s,t)\right]ds. \tag{4.30}$$

This completes our BSVIE-approach and we proceed to the BSDE-based setting. Ankirchner et al. [2] derived a differentiability result for quadratic backward stochastic differential equations, which is similar to our result for BSVIEs. We will first state in short the result from [2]. Then we will apply the differentiability result from [2], in the same way as we have applied Theorem 4.1 (respectively Theorem 5.2) to the BSVIE-based setting. This will lead to the construction of continuous-time dynamic capital allocations that are based on BSDEs.

Similar to our more general approach in Section 5 Ankirchner et al. [2] start with parameter dependent BSDEs of the following form

$$Y^x(t) = \xi(x) + \int_t^T [l(s,x,Y(s),Z(s)) + \alpha|Z^x(s)|^2]ds - \int_t^T Z^x(s)dW(s), \tag{4.31}$$

for $x\in\mathbb{R}^n$, where l is Lipschitz-continuous. They derive under similar conditions as (D1) and (D2) from Section 5 that $x\mapsto (Y^x(\cdot),Z^x(\cdot))$ is differentiable, and the derivative is a solution to the BSDE

$$\begin{aligned}\nabla Y^x(t) = \nabla\xi(x) &- \int_t^T \nabla Z^x(s)dW(s)\\ &+ \int_t^T [\partial_x l(s,x,Y^x(s),Z^x(s)) + \partial_y l(s,x,Y^x(s),Z^x(s))\nabla Y^x(s)\\ &+ \partial_z l(s,x,Y^x(s),Z^x(s))\nabla Z^x(s) + 2\alpha Z^x(s)\nabla Z^x(s)]ds.\end{aligned} \tag{4.32}$$

The results on BSDE-based dynamic risk measures and the differentiability result from above enable us to define the BSDE-version of the dynamic gradient allocation as

$$\frac{d}{du}\rho(t;\xi+u\eta)\bigg|_{u=0} = \nabla Y^u(t)|_{u=0} =: \nabla Y(t). \tag{4.33}$$

With the result from [2], $\frac{d}{du}\rho(t;\xi+u\eta))|_{u=0} = \nabla Y(t)$ exists as the first component of the unique adapted solution to the BSDE

$$\begin{aligned}\nabla Y(t) = -\eta &- \int_t^T \nabla Z(s)dW(s)\\ &+ \int_t^T [\partial_z l(s,Z(s)) + 2\alpha Z(s)]\nabla Z(s)ds.\end{aligned} \tag{4.34}$$

The above results enable us to study the dynamic entropic risk measure and to derive the corresponding dynamic gradient allocation.

Example 4.5. *The dynamic entropic risk measure is defined as*

$$e_\gamma(t;\xi) = \gamma\ln\mathbb{E}\left[e^{-\frac{1}{\gamma}\xi}\,\middle|\,\mathscr{F}_t\right], \quad t\in[0,T]. \tag{4.35}$$

According to Proposition 6.4 in Barrieu and El Karoui [4] $(e_\gamma(t;\xi); t \in [0,T])$ *is the first component of the solution of the following BSDE with quadratic generator* $g(t,z) = \frac{1}{2\gamma}|z|^2$ *and bounded and* $\mathscr{F}_T$*-measurable terminal condition* ξ*,*

$$e_\gamma(t;\xi) = -\xi - \int_t^T Z(s)dW(s) + \int_t^T \frac{1}{2\gamma}|Z(s)|^2 ds \tag{4.36}$$

This follows from a multiplicative decomposition of $\mathbb{E}[\exp -(1/\gamma)\xi|\mathscr{F}_t]$ *and the application of Itô's formula to the function* $\gamma \ln(x)$*.*

The driver $g(t,z) = \frac{1}{2\gamma}|z|^2$ *satisfies the regularity conditions from [2] which allows us to apply the results provided by* (4.33) *and* (4.34) *to the dynamic entropic risk measure, respectively to the BSDE* (4.36)*. It follows that the dynamic gradient allocation with respect to the dynamic entropic risk measure is given by*

$$\left.\frac{d}{du} e_\gamma(t;\xi + u\eta)\right|_{u=0} = \nabla Y^u(t)|_{u=0} =: \nabla Y(t),$$

where

$$\nabla Y(t) = \left.\frac{d}{du} e_\gamma(t;\xi + u\eta)\right|_{u=0} = \frac{\mathbb{E}[-\eta e^{-\frac{1}{\gamma}\xi}|\mathscr{F}_t]}{\mathbb{E}[e^{-\frac{1}{\gamma}\xi}|\mathscr{F}_t]}$$

is the first component of the unique solution to the BSDE

$$\nabla Y(t) = -\eta - \int_t^T \nabla Z(s)dW(s) + \int_t^T \frac{1}{\gamma} Z(s)\nabla Z(s)ds \tag{4.37}$$

with a bounded and $\mathscr{F}_T$*-measurable random variable* η*. Note that* $Z(\cdot)$ *is the second component of the solution to the BSDE* (4.36)*. Now define*

$$M(t) = \mathbb{E}\left[e^{-\frac{1}{\gamma}\xi}\middle|\mathscr{F}_t\right]$$

for $\xi \in L^\infty(\mathscr{F}_T)$*. Then* $M(\cdot)$ *is a positive bounded continuous martingale that has the multiplicative decomposition (see Proposition 6.4 in [4])*

$$M(t) = M(0) + \frac{1}{\gamma}\int_0^t M(s)Z^\xi(s)dW(s), \tag{4.38}$$

where $Z^\xi(\cdot)$ *satisfies* $\mathbb{E}[\int_t^T |Z^\xi(s)|^2 ds|\mathscr{F}_t] < \infty$ *for any* $0 \le t \le T$ *and thus is from* $BMO(\mathbb{P})$*. Moreover,* $M(\cdot)$ *satisfies*

$$M(t) = M(0)\exp\left\{\frac{1}{\gamma}\int_0^t Z^\xi(s)dW(s) - \frac{1}{2\gamma^2}\int_0^t |Z^\xi|^2 ds\right\}.$$

Define $L(t) = M(t)/M(0)$*. Since* $Z^\xi(\cdot)$ *is from* $BMO(\mathbb{P})$ *the Kazamaki Theorem (see Theorem 7.2 in [4]) allows us to define an equivalent probability measure* Q^ξ *by*

$$\left.\frac{dQ^\xi}{d\mathbb{P}}\right|_{\mathscr{F}_t} = L(t).$$

Since $W(\cdot)$ *is a Brownian motion under* $\mathbb{P}$, $\overline{W}(t) = W(t) - \frac{1}{\gamma}\int_0^t Z^\xi ds$ *is a Brownian motion under* Q^ξ*. Define* $H(t) = \mathbb{E}_{Q^\xi}[-\eta|\mathscr{F}_t]$ *for* $\eta \in L^\infty(\mathscr{F}_T)$*. Then by the martingale representation theorem we get*

$$H(t) = H(T) - \int_t^T Z^\eta(s)d\overline{W}(s), \tag{4.39}$$

and we immediately see that under $\mathbb{P}$ $H(\cdot)$ satisfies

$$H(t) = -\eta - \int_t^T Z^\eta(s)dW(s) + \frac{1}{\gamma}\int_t^T Z^\eta(s)Z^\xi(s)ds.$$

Furthermore we know that

$$\begin{aligned} H(t) = \mathbb{E}_{Q^\xi}[-\eta|\mathscr{F}_t] &= \frac{1}{L(t)}\mathbb{E}_\mathbb{P}[-\eta L(T)|\mathscr{F}_t] \\ &= \frac{M(0)}{M(t)}\mathbb{E}_\mathbb{P}\left[-\eta\frac{M(T)}{M(0)}\middle|\mathscr{F}_t\right] \\ &= \frac{1}{M(t)}\mathbb{E}_\mathbb{P}[-\eta M(T)|\mathscr{F}_t] \\ &= \frac{\mathbb{E}_\mathbb{P}[-\eta e^{-\frac{1}{\gamma}\xi}|\mathscr{F}_t]}{\mathbb{E}_\mathbb{P}[e^{-\frac{1}{\gamma}\xi}|\mathscr{F}_t]} \end{aligned}$$

From this we see that the dynamic gradient allocation with respect to the dynamic entropic risk measure at portfolio ξ in direction of the subportfolio η is the first component of the solution to the BSDE

$$Y(t) = -\eta - \int_t^T Z^\eta(s)dW(s) + \frac{1}{\gamma}\int_t^T Z^\eta(s)Z^\xi(s)ds$$

and we can identify the processes $Z(\cdot)$ and $\nabla Z(\cdot)$ from the general result in this case with the processes $Z^\eta(\cdot)$ and $Z^\xi(\cdot)$ from equations (4.38) *and* (4.39).

5 Differentiability of BSVIEs

For our purpose of deriving a classical differentiability result for BSVIEs, we need to study linear BSVIEs with a stochastic Lipschitz continuous generator. For our convenience Theorem 2.2 already covers these types of BSVIEs under appropriate conditions. These linear BSVIEs arise from formally differentiating BSVIEs of type (2.2) that depend on the Euclidean parameter set $\mathbb{R}^n$ for some $n \in \mathbb{N}$, namely

$$Y^x(t) = \psi(x,t) - \int_t^T Z^x(t,s)dW(s) + \int_t^T g(t,s,x,Y^x(s),Z^x(t,s),Z^x(s,t))ds, \tag{5.40}$$

for any $t \in [0,T]$ and $x \in \mathbb{R}^n$.

Now consider the following linear BSVIE

$$Y(t) = \psi(t) - \int_t^T Z(t,s)dW(s) + \int_t^T [m(t,s,Y(s),Z(t,s),Z(s,t)) + A(t,s)]ds, \tag{5.41}$$

where $m : \Delta^c \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d} \times \Omega \to \mathbb{R}^m$ and $\psi : [0,T] \times \Omega \to \mathbb{R}^m$ are given maps. We will make the following assumptions concerning the drivers.

(H2) Let $m : \Delta^c \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d} \times \Omega \to \mathbb{R}^m$ be $\mathscr{B}(\Delta^c \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d}) \otimes \mathscr{F}_T$-measurable such that $s \mapsto m(t,s,y,z,\zeta)$ is $\mathbb{F}$-progressively measurable for all $(t,y,z,\zeta) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d}$. Furthermore let m be linear in (y,z,ζ) (such that $m(t,s,0,0,0) = 0$ a.s.) and such that

$$|m(\omega,t,s,y,z,\zeta)| \leq L(t,s)(|y| + |z| + |\zeta|)$$

for a deterministic function $L : \Delta^c \to [0,\infty)$ with property (2.11) for some $\varepsilon > 0$.

Moreover let $A(\cdot,\cdot)$ be such that $A(t,\cdot)$ is $\mathbb{F}$-adapted for almost all $t \in [0,T]$ and

$$\mathbb{E}\left[\int_0^T \left(\int_0^T |A(t,s)|ds\right)^2 dt\right] < \infty. \tag{5.42}$$

Since condition (H2) is stronger than (H1) we can formulate the following Proposition as a direct consequence of Theorem 2.2.

Proposition 5.1. *Let (H2) hold. Then for any $\psi(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$ BSVIE* (5.41) *admits a unique adapted M-solution $(Y(\cdot),Z(\cdot,\cdot)) \in \mathscr{H}^2[0,T]$ on $[0,T]$. Moreover, the following estimate holds:*

$$||(Y(\cdot),Z(\cdot,\cdot))||^2_{\mathscr{H}^2[S,T]} \leq C\mathbb{E}\left[\int_S^T |\psi(t)|^2 dt + \int_S^T \left(\int_t^T |A(t,s)|ds\right)^2 dt\right] \tag{5.43}$$

Proof. The statement follows directly from Theorem 2.2. □

Another useful estimate from Yong [34] is the stability estimate. Let $\psi(\cdot)$, $\overline{\psi}(\cdot) \in L^2_{\mathscr{F}_T}(0,T)$ and $(Y(\cdot),Z(\cdot,\cdot)),(\overline{Y}(\cdot),\overline{Z}(\cdot,\cdot)) \in \mathscr{H}^2[0,T]$ be the adapted solutions of (2.2) with respect to g and $\psi(\cdot)$ respectively $\overline{g}$ and $\overline{\psi}(\cdot)$. Define $\delta Y(t) = Y(t) - \overline{Y}(t)$, $\delta Z(t,s) = Z(t,s) - \overline{Z}(t,s)$, $\delta\psi(t) = \psi(t) - \overline{\psi}(t)$ and

$$\delta g(t,s) = g(t,s,Y(s),Z(t,s),Z(s,t)) - \overline{g}(t,s,Y(s),Z(t,s),Z(s,t)).$$

Then the stability estimate from [34] states that

$$\begin{aligned}||(\delta Y(\cdot),\delta Z(\cdot,\cdot))||^2_{\mathscr{H}^2[S,T]} \leq C\mathbb{E}\Bigg[&\int_S^T |\delta\psi(t)|^2 dt \\ &+ \int_S^T \left(\int_t^T |\delta g(t,s)|\right)^2 dt\Bigg]\end{aligned} \tag{5.44}$$

for all $S \in [0,T]$.

This estimate will be important for our purposes. Therefore, and for the sake of completion, we elaborate here how to apply Theorem 2.2 (or the techniques behind Theorem 2.2) to obtain this estimate. Define for $i = 1,\dots,d$

$$\begin{aligned}\check{Z}^{(i)}(t,s) &= (\overline{Z}_1(t,s),\dots,\overline{Z}_i(t,s),Z_{i+1}(t,s),\dots,Z_d(t,s)),\\ \check{Z}^{(i)}(s,t) &= (\overline{Z}_1(s,t),\dots,\overline{Z}_i(s,t),Z_{i+1}(s,t),\dots,Z_d(s,t)),\end{aligned}$$

and

$$\overline{g}_{z_i}(t,s) = \Bigg[\frac{\overline{g}(t,s,\overline{Y}(s),\check{Z}^{(i-1)}(t,s),Z(s,t))\delta Z_i(t,s)^T}{|\delta Z_i(t,s)|^2} - \frac{\overline{g}(t,s,\overline{Y}(s),\check{Z}^{(i)}(t,s),Z(s,t))\delta Z_i(t,s)^T}{|\delta Z_i(t,s)|^2}\Bigg]1_{\{\delta Z_i(t,s)\neq 0\}},$$

$$\overline{g}_{\zeta_i}(t,s) = \Bigg[\frac{\overline{g}(t,s,\overline{Y}(s),\overline{Z}(t,s),\check{Z}^{(i-1)}(s,t))\delta Z_i(s,t)^T}{|\delta Z_i(s,t)|^2} - \frac{\overline{g}(t,s,\overline{Y}(s),\overline{Z}(t,s),\check{Z}^{(i)}(s,t))\delta Z_i(s,t)^T}{|\delta Z_i(s,t)|^2}\Bigg]1_{\{\delta Z_i(s,t)\neq 0\}},$$

$$\overline{g}_{y}(t,s) = \Bigg[\frac{\overline{g}(t,s,Y(s),Z(t,s),Z(s,t))\delta Y(s)^T}{|\delta Y(s)|^2} - \frac{\overline{g}(t,s,\overline{Y}(s),Z(t,s),Z(s,t))\delta Y(s)^T}{|\delta Y(s)|^2}\Bigg]1_{\{\delta Y(s)\neq 0\}},$$

Then

$$\begin{aligned}
&\sum_{i=1}^d \int_t^T \overline{g}_{z_i}(t,s)\delta Z_i(t,s)ds \\
&= \sum_{i=1}^d \int_t^T [\overline{g}(t,s,\overline{Y}(s),\check{Z}^{(i-1)}(t,s),Z(s,t)) - \overline{g}(t,s,\overline{Y}(s),\check{Z}^{(i)}(t,s),Z(s,t))]ds \\
&= \int_t^T [\overline{g}(t,s,\overline{Y}(s),Z(t,s),Z(s,t)) - \overline{g}(t,s,\overline{Y}(s),\overline{Z}(t,s),Z(s,t))]ds
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^d \int_t^T \overline{g}_{\zeta_i}(t,s)\delta Z_i(s,t)ds \\
&= \sum_{i=1}^d \int_t^T [\overline{g}(t,s,\overline{Y}(s),\overline{Z}(t,s),\check{Z}^{(i-1)}(s,t)) - \overline{g}(t,s,\overline{Y}(s),\overline{Z}(t,s),\check{Z}^{(i)}(s,t))]ds \\
&= \int_t^T [\overline{g}(t,s,\overline{Y}(s),\overline{Z}(t,s),Z(s,t)) - \overline{g}(t,s,\overline{Y}(s),\overline{Z}(t,s),\overline{Z}(s,t))]ds.
\end{aligned}$$

This leads to

$$\begin{aligned}
\delta Y(t) &= \delta\psi(t) + \int_t^T [g(t,s,Y(s),Z(t,s),Z(s,t)) - \overline{g}(t,s,\overline{Y}(s),\overline{Z}(t,s),\overline{Z}(s,t))]ds \\
&\quad - \int_t^T \delta Z(t,s)dW(s) \\
&= \delta\psi(t) + \int_t^T \Big(\overline{g}_y(t,s)\delta Y(s) + \sum_{i=1}^d \big[\overline{g}_{z_i}(t,s)\delta Z_i(t,s) + \overline{g}_{\zeta_i}(t,s)\delta Z_i(s,t)\big] \\
&\quad + \delta g(t,s)\Big)ds - \int_t^T \delta Z(t,s)dW(s)
\end{aligned} \tag{5.45}$$

The driver in the above BSVIE is of the same type as in the linear BSVIE (5.41) and since g satisfies (H1) the driver of (5.45) satisfies (H2) such that the application of Theorem 2.2 respectively

Proposition 5.1 leads to the existence and uniqueness of M-solutions to BSVIE (5.45) and to the stability estimate (5.44). This result obviously applies directly to solutions to the linear BSVIE (5.41) under condition (H2) such that the corresponding stability estimate is

$$||(\delta Y(\cdot),\delta Z(\cdot,\cdot))||^2_{\mathscr{H}^2[S,T]} \leq C\mathbb{E}\left[\int_S^T |\delta\psi(t)|^2 dt \right. \\ \left. + \int_S^T \left(\int_t^T |\delta A(t,s)|\right)^2 dt\right] \tag{5.46}$$

for all $S \in [0,T]$.

With these results we are now able to state and prove our differentiability result for parameter dependent BSVIE of type (5.40). As before, we will first make several assumptions concerning the drivers.

(D1) Let $g : \Delta^c \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d} \times \Omega \to \mathbb{R}^m$ be $\mathscr{B}(\Delta^c \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d}) \otimes \mathscr{F}_T$-measurable such that $s \mapsto g(t,s,x,y,z,\zeta)$ is $\mathbb{F}$-progressively measurable for all $(t,x,y,z,\zeta) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d}$ and

$$\mathbb{E}\left[\int_0^T \left(\int_0^T |g(t,s,x,0,0,0)|ds\right)^2 dt\right] < \infty \quad \forall x \in \mathbb{R}^n. \tag{5.47}$$

Moreover let g be such that $(x,y,z,\zeta) \mapsto g(t,s,x,y,z,\zeta)$ is continuously differentiable and there exists a deterministic function $L : \Delta^c \to [0,\infty)$ satisfying (2.11) such that

$$\left|\frac{\partial}{\partial y} g(t,s,x,y,z,\zeta)\right| \leq L(t,s), \tag{5.48}$$

$$\left|\sum_{j=1}^d \frac{\partial}{\partial z_j} g(t,s,x,y,z,\zeta)\eta_j\right| \leq L(t,s)|\eta|, \tag{5.49}$$

$$\left|\sum_{j=1}^d \frac{\partial}{\partial \zeta_j} g(t,s,x,y,z,\zeta)\eta_j\right| \leq L(t,s)|\eta|, \tag{5.50}$$

$$\tag{5.51}$$

for all $(t,s) \in \Delta^c$, $(x,y,z,\zeta) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d}$, $\eta \equiv (\eta_1,\ldots,\eta_d) \in \mathbb{R}^{m\times d}$, a.s. Furthermore let $x \mapsto g(t,s,x,y,z,\zeta)$ be differentiable and for all $x \in \mathbb{R}^n$

$$\lim_{x\to x'} \mathbb{E}\left[\int_0^T \left(\int_0^T \left|\frac{\partial}{\partial x} g(t,s,x',Y^{x'}(s),Z^{x'}(t,s),Z^{x'}(s,t)) \right.\right.\right. \\ \left.\left.\left. - \frac{\partial}{\partial x} g(t,s,x,Y^x(s),Z^x(t,s),Z^x(s,t))\right| ds\right)^2 dt\right] = 0 \tag{5.52}$$

(D2) $\psi : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^m$, $\psi(t,x) \in L^2_{\mathscr{F}_T}(0,T)$, $x \mapsto \psi(t,x)$ is differentiable for all $t \in [0,T]$ with derivative $\nabla\psi(t,x) \in L^2_{\mathscr{F}_T}(0,T)$ for any $x \in \mathbb{R}^n$.

Note that, as already mentioned in Yong [34], the conditions (5.48)-(5.50) imply the Lipschitz condition for the map $(y,z,\zeta) \mapsto g(t,s,x,y,z,\zeta)$ with the Lipschitz constants depending on (t,s). Thus (D1) is stronger than (H1).

With these assumptions our differentiability result follows. This result is the main contribution of the paper. It allows us to construct the dynamic gradient allocation from solutions to BSVIEs. For applications in the previous sections we have only needed the one-dimensional case and a generator that was independent of $Y(\cdot)$ or only linear in $Y(\cdot)$. Here we consider the general situation.

Theorem 5.2. *Let (D1) and (D2) hold. Then the function*

$$\mathbb{R}^n \to L^2_{\mathbb{F}}(0,T) \times L^2_{\mathbb{F}}(0,T;L^2_{\mathbb{F}}(0,T)), \quad x \mapsto (Y^x(\cdot), Z^x(\cdot,\cdot))$$

is differentiable and the derivative is the unique adapted M-solution to the BSVIE

$$\begin{aligned}
&\nabla Y^x(t) \\
&= \nabla \psi(t,x) - \int_t^T \nabla Z^x(t,s) dW(s) \\
&+ \int_t^T \Big[\partial_x g(t,s,x,Y^x(s),Z^x(t,s),Z^x(s,t)) + \partial_y g(t,s,x,Y^x(s),Z^x(t,s),Z^x(s,t)) \nabla Y^x(s) \\
&+ \sum_{j=1}^d \Big[\frac{\partial}{\partial z_j} g(t,s,x,Y^x(s),Z^x(t,s),Z^x(s,t)) \nabla Z^x_j(t,s) \\
&+ \frac{\partial}{\partial \zeta_j} g(t,s,x,Y^x(s),Z^x(t,s),Z^x(s,t)) \nabla Z^x_j(s,t) \Big] \Big] ds
\end{aligned} \tag{5.53}$$

Proof. Let $\mathscr{Y}^h(t) = \frac{1}{h}(Y^{x+e_ih}(t) - Y^x(t))$, $\mathscr{Z}^h(t,s) = \frac{1}{h}(Z^{x+e_ih}(t,s) - Z^x(t,s))$, $\mathscr{Z}^h(s,t) = \frac{1}{h}(Z^{x+e_ih}(s,t) - Z^x(s,t))$ and $\Psi^h(t) = \frac{1}{h}(\psi(x+e_ih,t) - \psi(x,t))$ for any $t,s \in [0,T]$. Then from (5.40) it follows for any $h \neq 0$

$$\begin{aligned}
\mathscr{Y}^h(t) = \Psi^h(t) &- \int_t^T \mathscr{Z}^h(t,s) dW(s) \\
&+ \int_t^T \frac{1}{h} \Big[g(t,s,x+e_ih, Y^{x+e_ih}(s), Z^{x+e_ih}(t,s), Z^{x+e_ih}(s,t)) \\
&\qquad - g(t,s,x,Y^x(s),Z^x(t,s),Z^x(s,t)) \Big] ds.
\end{aligned} \tag{5.54}$$

Define $l_{x,h} : [0,1] \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d}$,

$$\begin{aligned}
l_{x,h}(\theta) = \big(& x + \theta e_i h, Y^x(s) + \theta [Y^{x+e_ih}(s) - Y^x(s)], \\
& Z^x(t,s) + \theta [Z^{x+e_ih}(t,s) - Z^x(t,s)], Z^x(s,t) + \theta [Z^{x+e_ih}(s,t) - Z^x(s,t)] \big).
\end{aligned}$$

With the above notation it follows

$$\frac{1}{h} l'_{x,h}(\theta) = (e_i, \mathscr{Y}^h(s), \mathscr{Z}^h(t,s), \mathscr{Z}^h(s,t)).$$

Furthermore we get

$$\begin{aligned}
&\frac{1}{h} \Big[g(t,s,x+e_ih, Y^{x+e_ih}(s), Z^{x+e_ih}(t,s), Z^{x+e_ih}(s,t)) \\
&\quad - g(t,s,x,Y^x(s),Z^x(t,s),Z^x(s,t)) \Big] \\
&= \frac{1}{h} \Big[g(l^{x,h}(1)) - g(l^{x,h}(0)) \Big] = \frac{1}{h} \int_0^1 \frac{\partial}{\partial \theta} g(l^{x,h}(\theta)) d\theta
\end{aligned}$$

The term on the right in the previous equation can now be modified to

$$
\begin{aligned}
&\frac{1}{h}\int_0^1 \frac{\partial}{\partial\theta} g(l^{x,h}(\theta))d\theta = \int_0^1 \left\langle \nabla g(l^{x,h}(\theta)), \frac{1}{h} l'_{x,h}(\theta) \right\rangle d\theta \\
&= \int_0^1 \frac{\partial}{\partial x_i} g(l_{x,h}(\theta))d\theta + \int_0^1 \frac{\partial}{\partial y} g(l_{x,h}(\theta))d\theta \mathscr{Y}^h(t) \\
&\quad + \int_0^1 \frac{\partial}{\partial z} g(l_{x,h}(\theta))d\theta \mathscr{Z}^h(t,s) + \int_0^1 \frac{\partial}{\partial \zeta} g(l_{x,h}(\theta))d\theta \mathscr{Z}^h(s,t) \\
&= A_{x,h}(t,s) + B_{x,h}(t,s)\mathscr{Y}^h(s) \\
&\quad + C_{x,h}(t,s)\mathscr{Z}^h(t,s) + D_{x,h}(t,s)\mathscr{Z}^h(s,t)
\end{aligned}
$$

with

$$
\begin{aligned}
A_{x,h}(t,s) &= \int_0^1 \frac{\partial}{\partial x_i} g(l_{x,h}(\theta))d\theta \\
B_{x,h}(t,s) &= \int_0^1 \frac{\partial}{\partial y} g(l_{x,h}(\theta))d\theta \\
C_{x,h}(t,s) &= \int_0^1 \frac{\partial}{\partial z} g(l_{x,h}(\theta))d\theta \\
D_{x,h}(t,s) &= \int_0^1 \frac{\partial}{\partial \zeta} g(l_{x,h}(\theta))d\theta
\end{aligned}
$$

Now define

$$m^{s,t}_{x,h}(u,v,w) = (B_{x,h}(t,s)u + C_{x,h}(t,s)v + D_{x,h}(t,s)w)$$

and we obtain in these terms from (5.54)

$$
\begin{aligned}
\mathscr{Y}^h(t) &= \Psi^h(t) - \int_t^T \mathscr{Z}^h(t,s)dW(s) \\
&\int_t^T [m^{s,t}_{x,h}(\mathscr{Y}^h(s), \mathscr{Z}^h(t,s), \mathscr{Z}^h(s,t)) + A_{x,h}(t,s)]ds.
\end{aligned} \tag{5.55}
$$

Now, from (D1) condition (H2) is satisfied and it follows from (5.46)

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T |\mathscr{Y}^h(t) - \mathscr{Y}^{h'}(t)|^2 dt\right] \le C\Bigg(&\mathbb{E}\left[\int_0^T |\Psi^h(t) - \Psi^{h'}(t)|^2 dt\right] \\
&+ \mathbb{E}\left[\int_0^T \left(\int_t^T A_{x,h}(t,s) - A_{x,h'}(t,s)ds\right)^2 dt\right]\Bigg).
\end{aligned}
$$

From (D2) we get

$$\lim_{h,h'\to 0} \mathbb{E}\left[\int_0^T |\Psi^h(t) - \Psi^{h'}(t)|^2 dt\right] = 0,$$

and (D1) provides us with

$$\lim_{h,h'\to 0} \mathbb{E}\left[\int_0^T \left(\int_t^T A_{x,h}(t,s) - A_{x,h'}(t,s)ds\right)^2 dt\right] = 0$$

and thus finally

$$\lim_{h,h'\to 0} \mathbb{E}\left[\int_0^T |\mathscr{Y}^h(t) - \mathscr{Y}^{h'}(t)|^2 dt\right] = 0$$

follows. With the same arguments we get from (5.46) with (D1) and (D2)

$$\lim_{h,h'\to 0}\mathbb{E}\left[\int_0^T\int_0^T|\mathscr{Z}^h(t,s)-\mathscr{Z}^{h'}(t,s)|^2dsdt\right]=0,$$

which gives us

$$\lim_{h,h'\to 0}||(\mathscr{Y}^h(\cdot)-\mathscr{Y}^{h'}(\cdot),\mathscr{Z}^h(t,s)-\mathscr{Z}^{h'}(t,s))||_{\mathscr{H}^2[0,T]}=0.$$

Now let $(h_n)_n$ be a sequence in $\mathbb{R}\setminus\{0\}$ converging to zero. Since $L^2_{\mathbb{F}}(0,T)$ and $L^2(0,T;L^2_{\mathbb{F}}(0,T))$ are Banach spaces the sequence $\mathscr{Y}^{h_n}(t)$ converges to a process $\frac{\partial}{\partial x_i}Y^x(t)$ and $\mathscr{Z}^{h_n}(t,s)$ to a process $\frac{\partial}{\partial x_i}Z^x(t,s)$ with respect to the corresponding norms. The term by term convergence of the difference quotient version of our BSVIE and its formal derivative gives us that $(\frac{\partial}{\partial x_i}Y^x(\cdot),\frac{\partial}{\partial x_i}\mathscr{Z}^x(\cdot,\cdot))$ is a solution to the BSVIE

$$\begin{aligned}
&\frac{\partial}{\partial x_i}Y^x(t)\\
&=\frac{\partial}{\partial x_i}\psi(t,x)-\int_t^T\frac{\partial}{\partial x_i}Z^x(t,s)dW(s)\\
&\quad+\int_t^T\Big[\frac{\partial}{\partial x_i}g(t,s,x,Y^x(s),Z^x(t,s),Z^x(s,t))\\
&\qquad+\frac{\partial}{\partial y}g(t,s,x,Y^x(s),Z^x(t,s),Z^x(s,t))\frac{\partial}{\partial x_i}Y^x(s)\\
&\qquad\sum_{j=1}^d\Big[\frac{\partial}{\partial z_j}g(t,s,x,Y^x(s),Z^x(t,s),Z^x(s,t))\frac{\partial}{\partial x_i}Z_j^x(t,s)\\
&\qquad+\frac{\partial}{\partial\zeta_j}g(t,s,x,Y^x(s),Z^x(t,s),Z^x(s,t))\frac{\partial}{\partial x_i}Z_j^x(s,t)\Big]\Big]ds.
\end{aligned}$$

It follows from the first part of our proof that

$$\lim_{h\to 0}\left|\left|\mathscr{Y}^h(\cdot)-\frac{\partial}{\partial x_i}Y^x(\cdot),\mathscr{Z}^h(\cdot,\cdot)-\frac{\partial}{\partial x_i}\mathscr{Z}^x(\cdot,\cdot)\right|\right|_{\mathscr{H}^2[0,T]}=0,$$

and thus $x\mapsto(Y^x(\cdot),Z^x(\cdot,\cdot))$ is partially differentiable. Now denote

$$\nabla Y^x(\cdot)=(\partial/\partial_{x_1}Y^x(\cdot),\dots,\partial/\partial_{x_n}Y^x(\cdot)).$$

The stability estimate (5.46) implies that $x\mapsto(\nabla Y^x(\cdot),\nabla Z^x(\cdot,\cdot))$ is continuous (w.r.t. the norm $||\,.\,||_{\mathscr{H}^2[0,T]}$) and hence $(Y^x(\cdot),Z^x(\cdot,\cdot))$ is totally differentiable and the result follows. □

6 Conclusion

We study the capital allocation problem in a continuous-time dynamic risk setting with risk measures that arise as solutions to specific types of backward stochastic differential equations or backward stochastic Volterra integral equations. For these classes of risk measures we provide the corresponding dynamic gradient allocations as solutions to specific linear BSVIEs or BSDEs. This is enabled by a general classical differentiability result for BSVIEs. The proof of this result rounds off the paper and is presented in the final section. The results are applied to an example of a BSVIE-based dynamic risk measure from Yong [33] and to the dynamic entropic risk measure. For this specific dynamic risk measure we furthermore derive a representation of the dynamic gradient allocation in terms of a conditional expectation under an equivalent probability measure Q that is constructed from the martingale representation of the underlying entropic risk measure.

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